Digraphs

Version 1.7.1

Jan De Beule
Julius Jonušas
James D. Mitchell
Maria Tsalakou
Wilf A. Wilson
Michael C. Young
Marina Anagnostopoulou-Merkouri
Finn Buck
Stuart Burrell
Reinis Cirpons
Tom Conti-Leslie
Luke Elliott
Ewan Gilligan
Max Horn
Christopher Jefferson
Markus Pfeiffer
Lea Racine
Christopher Russell
Finn Smith
Ben Spiers
Murray White
Jan De Beule  Email: jdebeule@cage.ugent.be  
Homepage: http://homepages.vub.ac.be/~jdebeule

Julius Jonušas  Email: julius.jonusas@tuwien.ac.at  
Homepage: http://julius.jonusas.work

James D. Mitchell  Email: jdm3@st-and.ac.uk  
Homepage: https://jdbm.me

Maria Tsalakou  Email: mt200@st-andrews.ac.uk  
Homepage: https://mariatsalakou.github.io

Wilf A. Wilson  Email: gap@wilf-wilson.net  
Homepage: https://wilf.me

Michael C. Young  Email: mct25@st-andrews.ac.uk  
Homepage: https://mct25.host.cs.st-andrews.ac.uk
Abstract

The Digraphs package is a GAP package containing methods for graphs, digraphs, and multidigraphs.

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Acknowledgements

We would like to thank Christopher Jefferson for his help in including bliss in Digraphs. This package’s methods for computing digraph homomorphisms are based on work by Max Neunhöffer, and independently Artur Schäfer.
## Contents

1 The Digraphs package ................................................................. 5
  1.1 Introduction ........................................................................... 5

2 Installing Digraphs ................................................................... 8
  2.1 For those in a hurry ............................................................... 8
  2.2 Optional package dependencies .......................................... 9
  2.3 Compiling the kernel module .............................................. 10
  2.4 Rebuilding the documentation ............................................ 10
  2.5 Testing your installation .................................................... 10

3 Creating digraphs .................................................................... 12
  3.1 Creating digraphs ................................................................. 12
  3.2 Changing representations .................................................. 19
  3.3 New digraphs from old ....................................................... 22
  3.4 Random digraphs ............................................................... 50
  3.5 Standard examples ............................................................ 52

4 Operators ................................................................................. 79
  4.1 Operators for digraphs ....................................................... 79

5 Attributes and operations ......................................................... 82
  5.1 Vertices and edges .............................................................. 82
  5.2 Neighbours and degree ..................................................... 91
  5.3 Orders ................................................................................. 101
  5.4 Reachability and connectivity ........................................... 102
  5.5 Cayley graphs of groups ....................................................... 123
  5.6 Associated semigroups ....................................................... 124
  5.7 Planarity .............................................................................. 126

6 Properties of digraphs .............................................................. 131
  6.1 Vertex properties ............................................................. 131
  6.2 Edge properties ............................................................... 132
  6.3 Edge Weights .................................................................... 140
  6.4 Orders ................................................................................. 141
  6.5 Regularity ......................................................................... 144
  6.6 Connectivity and cycles ...................................................... 146
  6.7 Planarity ............................................................................. 154
6.8 Homomorphisms and transformations ............................................. 155

7 Homomorphisms ................................................................. 157
   7.1 Acting on digraphs ......................................................... 157
   7.2 Isomorphisms and canonical labellings .................................. 159
   7.3 Homomorphisms of digraphs .............................................. 175

8 Finding cliques and independent sets ....................................... 189
   8.1 Finding cliques ............................................................. 190
   8.2 Finding independent sets ................................................ 196

9 Visualising and IO .............................................................. 200
   9.1 Visualising a digraph ....................................................... 200
   9.2 Reading and writing digraphs to a file ................................ 209

A Grape to Digraphs Command Map .......................................... 221
   A.1 Functions to construct and modify graphs .......................... 221
   A.2 Functions to inspect graphs, vertices and edges .................. 221
   A.3 Functions to determine regularity properties of graphs .......... 222
   A.4 Some special vertex subsets of a graph ............................. 222
   A.5 Functions to construct new graphs from old ....................... 223
   A.6 Vertex-Colouring and Complete Subgraphs ......................... 223
   A.7 Automorphism groups and isomorphism testing for graphs ...... 223

References ................................................................. 224

Index ................................................................. 225
Chapter 1

The Digraphs package

1.1 Introduction

This is the manual for version 1.7.1 of the Digraphs package. This package was developed at the University of St Andrews by:

- Jan De Beule,
- Julius Jonušas,
- James D. Mitchell,
- Maria Tsalakou,
- Wilf A. Wilson, and
- Michael C. Young.

Additional contributions were made by:

- Marina Anagnostopoulou-Merkouri,
- Finn Buck,
- Stuart Burrell,
- Reinis Cirpons,
- Tom Conti-Leslie,
- Luke Elliott,
- Ewan Gilligan,
- Max Horn,
- Christopher Jefferson,
- Markus Pfeiffer,
- Lea Racine,
The Digraphs package contains a variety of methods for efficiently creating and storing mutable and immutable digraphs and computing information about them. Full explanations of all the functions contained in the package are provided below.

If the Grape package is available, it will be loaded automatically. Digraphs created with the Digraphs package can be converted to Grape graphs with Graph (3.2.3), and conversely Grape graphs can be converted to Digraphs objects with Digraph (3.1.7). Grape is not required for Digraphs to run.

The bliss tool [JK07] is included in this package. It is an open-source tool for computing automorphism groups and canonical forms of graphs, written by Tommi Junttila and Petteri Kaski. Several of the methods in the Digraphs package rely on bliss. If the NautyTracesInterface package for GAP is available then it is also possible to use nauty [MP14] for computing automorphism groups and canonical forms in Digraphs. See Section 7.2 for more details.

The edge-addition-planarity-suite is also included in Digraphs; see [BM04], [Boy06], [BM06], and [Boy12]. The edge-addition-planarity-suite is an open-source implementation of the edge addition planar graph embedding algorithm and related algorithms by John M. Boyer. See Section 6.7 for more details.

From version 1.0.0 of this package, digraphs can be either mutable or immutable. Mutable digraphs can be changed in-place by many of the methods in the package, which avoids unnecessary copying. Immutable digraphs cannot be changed in-place, but their advantage is that the value of an attribute of an immutable digraph is only ever computed once. Mutable digraphs can be converted into immutable digraphs in-place using MakeImmutable (Reference: MakeImmutable). One of the motivations for introducing mutable digraphs in version 1.0.0 was that in practice the authors often wanted to create a digraph and immediately modify it (removing certain edges, loops, and so on). Before version 1.0.0, this involved copying the digraph several times, with each copy being discarded almost immediately. From version 1.0.0, this unnecessary copying can be eliminated by first creating a mutable digraph, then changing it in-place, and finally converting the mutable digraph to an immutable one in-place (if desirable).

1.1.1 Definitions

For the purposes of this package and its documentation, the following definitions apply:

A digraph $E = (E^0, E^1, r, s)$, also known as a directed graph, consists of a set of vertices $E^0$ and a set of edges $E^1$ together with functions $s, r : E^1 \rightarrow E^0$, called the source and range, respectively. The source and range of an edge is respectively the values of $s, r$ at that edge. An edge is called a loop if its source and range are the same. A digraph is called a multidigraph if there exist two or more edges with the same source and the same range.

A directed walk on a digraph is a sequence of alternating vertices and edges $$(v_1, e_1, v_2, e_2, ..., e_{n-1}, v_n)$$ such that each edge $e_i$ has source $v_i$ and range $v_{i+1}$. A directed path is a directed walk where no vertex (and hence no edge) is repeated. A directed circuit is a directed walk where $v_1 = v_n$, and a directed cycle is a directed circuit where where no vertex is repeated, except for $v_1 = v_n$. 

- Christopher Russell,
- Finn Smith,
- Ben Spiers, and
- Murray Whyte.
The length of a directed walk \((v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n)\) is equal to \(n - 1\), the number of edges it contains. A directed walk (or path) \((v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n)\) is sometimes called a directed walk (or path) from vertex \(v_1\) to vertex \(v_n\). A directed walk of zero length, i.e. a sequence \((v)\) for some vertex \(v\), is called trivial. A trivial directed walk is considered to be both a circuit and a cycle, as is the empty directed walk \(()\). A simple circuit is another name for a non-trivial and non-empty directed cycle.
Chapter 2

Installing Digraphs

2.1 For those in a hurry

In this section we give a brief description of how to start using Digraphs.

It is assumed that you have a working copy of GAP with version number 4.10.0 or higher. The most up-to-date version of GAP and instructions on how to install it can be obtained from the main GAP webpage https://www.gap-system.org.

The following is a summary of the steps that should lead to a successful installation of Digraphs:

- ensure that the IO package version 4.5.1 or higher is available. IO must be compiled before Digraphs can be loaded.

- ensure that the Orb package version 4.8.2 or higher is available. Orb has better performance when compiled, but although compilation is recommended, it is not required to be compiled for Digraphs to be loaded.

- ensure that the datastructures package version 0.2.5 or higher is available.

- This step is optional: certain functions in Digraphs require the Grape package to be available; see Section 2.2.1 for full details. To use these functions make sure that the Grape package version 4.8.1 or higher is available. If Grape is not available, then Digraphs can be used as normal with the exception that the functions listed in Subsection 2.2.1 will not work.

- This step is optional: certain functions in Digraphs require the NautyTracesInterface package to be available. If you want to make use of these functions, please ensure that the NautyTracesInterface package version 0.2 or higher is available. If NautyTracesInterface is not available, then Digraphs can be used as normal with the exception that functions whose names contain “Nauty” will not work.

- download the package archive digraphs-1.7.1.tar.gz from the Digraphs package webpage.

- unzip and untar the file, this should create a directory called digraphs-1.7.1.

- locate the pkg directory of your GAP directory, which contains the directories lib, doc and so on. Move the directory digraphs-1.7.1 into the pkg directory.

- it is necessary to compile the Digraphs package. Inside the pkg/digraphs-1.7.1 directory, type
Further information about this step can be found in Section 2.3.

- start GAP in the usual way (i.e. type `gap` at the command line).
- type `LoadPackage("digraphs");`

If you want to check that the package is working correctly, you should run some of the tests described in Section 2.5.

### 2.1.1 Configuration options

In addition to the usual autoconf generated configuration flags, the following flags are provided.

<table>
<thead>
<tr>
<th>Option</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>-enable-code-coverage</td>
<td>enable code coverage support</td>
</tr>
<tr>
<td>-enable-compile-warnings</td>
<td>enable compiler warnings</td>
</tr>
<tr>
<td>-enable-debug</td>
<td>enable debug mode</td>
</tr>
<tr>
<td>-with-external-bliss</td>
<td>use external bliss</td>
</tr>
<tr>
<td>-with-external-planarity</td>
<td>use external <code>edge-addition-planarity-suite</code></td>
</tr>
<tr>
<td>-with-gaproot</td>
<td>specify root of GAP installation</td>
</tr>
<tr>
<td>-without-intrinsics</td>
<td>do not use compiler intrinsics even if available</td>
</tr>
</tbody>
</table>

**Table:** Configuration flags

### 2.2 Optional package dependencies

The `Digraphs` package is written in GAP and C code and requires the `IO` package. The `IO` package is used to read and write transformations, partial permutations, and bipartitions to a file.

#### 2.2.1 The Grape package

The `Grape` package must be available for the following operations to be available:

- `Graph` (3.2.3) with a digraph argument
- `AsGraph` (3.2.4) with a digraph argument
- `Digraph` (3.1.7) with a `Grape` graph argument

If `Grape` is not available, then `Digraphs` can be used as normal with the exception that the functions above will not work.
2.3 Compiling the kernel module

The Digraphs package has a GAP kernel component in C which should be compiled. This component contains certain low-level functions required by Digraphs.

It is not possible to use the Digraphs package without compiling it.

To compile the kernel component inside the pkg/digraphs-1.7.1 directory, type

```
./configure
make
```

If you installed the package in another 'pkg' directory than the standard 'pkg' directory in your GAP installation, then you have to do two things. Firstly during compilation you have to use the option '--with-gaproot=PATH' of the 'configure' script where 'PATH' is a path to the main GAP root directory (if not given the default './.' is assumed).

If you installed GAP on several architectures, you must execute the configure/make step for each of the architectures. You can either do this immediately after configuring and compiling GAP itself on this architecture, or alternatively set the environment variable 'CONFIGNAME' to the name of the configuration you used when compiling GAP before running './configure'. Note however that your compiler choice and flags (environment variables 'CC' and 'CFLAGS') need to be chosen to match the setup of the original GAP compilation. For example you have to specify 32-bit or 64-bit mode correctly!

2.4 Rebuilding the documentation

The Digraphs package comes complete with pdf, html, and text versions of the documentation. However, you might find it necessary, at some point, to rebuild the documentation. To rebuild the documentation, please use the function DigraphsMakeDoc (2.4.1).

2.4.1 DigraphsMakeDoc

```
▷ DigraphsMakeDoc()

Returns: Nothing
```

This function should be called with no argument to compile the Digraphs documentation.

2.5 Testing your installation

In this section we describe how to test that Digraphs is working as intended. To test that Digraphs is installed correctly use DigraphsTestInstall (2.5.1) or for more extensive tests use DigraphsTestStandard (2.5.2).

If something goes wrong, then please review the instructions in Section 2.1 and ensure that Digraphs has been properly installed. If you continue having problems, please use the issue tracker to report the issues you are having.
2.5.1 DigraphsTestInstall

▷ DigraphsTestInstall() (function)

   Returns: true or false.

   This function can be called without arguments to test your installation of Digraphs is working correctly. These tests should take no more than a few seconds to complete. To test more comprehensively that Digraphs is working correctly, use DigraphsTestStandard (2.5.2).

2.5.2 DigraphsTestStandard

▷ DigraphsTestStandard() (function)

   Returns: true or false.

   This function can be called without arguments to test all of the methods included in Digraphs. These tests should take less than a minute to complete.

   To quickly test that Digraphs is installed correctly use DigraphsTestInstall (2.5.1). For a more thorough test, use DigraphsTestExtreme (2.5.3).

2.5.3 DigraphsTestExtreme

▷ DigraphsTestExtreme() (function)

   Returns: true or false.

   This function should be called with no argument. It executes a series of very demanding tests, which measure the performance of a variety of functions on large examples. These tests take a long time to complete, at least several minutes.

   For these tests to complete, the digraphs library digraphs-lib must be downloaded and placed in the digraphs directory in a subfolder named digraphs-lib. This library can be found on the Digraphs website.
Chapter 3

Creating digraphs

In this chapter we describe how to create digraphs.

3.1 Creating digraphs

3.1.1 IsDigraph

▶ IsDigraph

(Category)

Every digraph in Digraphs belongs to the category IsDigraph. Some basic attributes and operations for digraphs are DigraphVertices (5.1.1), DigraphEdges (5.1.3), and OutNeighbours (5.2.6).

3.1.2 IsMutableDigraph

▶ IsMutableDigraph

(Category)

IsMutableDigraph is a synonym for IsDigraph (3.1.1) and IsMutable (Reference: IsMutable). A mutable digraph may be changed in-place by methods in the Digraphs package, and is not attribute-storing – see IsAttributeStoringRep (Reference: IsAttributeStoringRep).

A mutable digraph may be converted into an immutable attribute-storing digraph by calling MakeImmutable (Reference: MakeImmutable) on the digraph.

3.1.3 IsImmutableDigraph

▶ IsImmutableDigraph

(Category)

IsImmutableDigraph is a subcategory of IsDigraph (3.1.1). Digraphs that lie in IsImmutableDigraph are immutable and attribute-storing. In particular, they lie in IsAttributeStoringRep (Reference: IsAttributeStoringRep).

A mutable digraph may be converted to an immutable digraph that lies in the category IsImmutableDigraph by calling MakeImmutable (Reference: MakeImmutable) on the digraph.

The operation DigraphMutableCopy (3.3.1) can be used to construct a mutable copy of an immutable digraph. It is however not possible to convert an immutable digraph into a mutable digraph in-place.
3.1.4 IsCayleyDigraph

\[ \text{IsCayleyDigraph} \]

IsCayleyDigraph is a subcategory of IsDigraph. Digraphs that are Cayley digraphs of a group and that are constructed by the operation CayleyDigraph (3.1.12) are constructed in this category, and are always immutable.

3.1.5 IsDigraphWithAdjacencyFunction

\[ \text{IsDigraphWithAdjacencyFunction} \]

IsDigraphWithAdjacencyFunction is a subcategory of IsDigraph. Digraphs that are created using an adjacency function are constructed in this category.

3.1.6 DigraphByOutNeighboursType

\[ \text{DigraphByOutNeighboursType} \]
\[ \text{DigraphFamily} \]

The type of all digraphs is DigraphByOutNeighboursType. The family of all digraphs is DigraphFamily.

3.1.7 Digraph

\[ \text{Digraph}([\text{filt}, \text{obj}], \text{source}, \text{range}) \]
\[ \text{Digraph}([\text{filt}, \text{list}, \text{func}) \]
\[ \text{Digraph}([\text{filt}, \text{G}, \text{list}, \text{act}, \text{adj}) \]

Returns: A digraph.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

for a list (i.e. an adjacency list)

if obj is a list of lists of positive integers in the range from 1 to Length(obj), then this function returns the digraph with vertices \( E^0 = [1 .. \text{Length}(obj)] \), and edges corresponding to the entries of obj.

More precisely, there is an edge from vertex \( i \) to \( j \) if and only if \( j \) is in \( \text{obj}[i] \); the source of this edge is \( i \) and the range is \( j \). If \( j \) occurs in \( \text{obj}[i] \) with multiplicity \( k \), then there are \( k \) edges from \( i \) to \( j \).

for three lists

if obj is a duplicate-free list, and source and range are lists of equal length consisting of positive integers in the list \( [1 .. \text{Length}(obj)] \), then this function returns a digraph with vertices \( E^0 = [1 .. \text{Length}(obj)] \), and Length(source) edges. For each \( i \) in \( [1
.. Length(source)] there exists an edge with source vertex source[i] and range vertex range[i]. See DigraphSource (5.2.5) and DigraphRange (5.2.5).

The vertices of the digraph will be labelled by the elements of obj.

for an integer, and two lists
if obj is an integer, and source and range are lists of equal length consisting of positive integers in the list [1 .. obj], then this function returns a digraph with vertices \( E^0 = [1 .. obj] \), and Length(source) edges. For each i in [1 .. Length(source)] there exists an edge with source vertex source[i] and range vertex range[i]. See DigraphSource (5.2.5) and DigraphRange (5.2.5).

for a list and a function
if list is a list and func is a function taking 2 arguments that are elements of list, and func returns true or false, then this operation creates a digraph with vertices [1 .. Length(list)] and an edge from vertex i to vertex j if and only if func(list[i], list[j]) returns true.

for a group, a list, and two functions
The arguments will be G, list, act, adj.

Let G be a group acting on the objects in list via the action act, and let adj be a function taking two objects from list as arguments and returning true or false. The function adj will describe the adjacency between objects from list, which is invariant under the action of G. This variant of the constructor returns a digraph with vertices the objects of list and directed edges [x, y] when \( f(x, y) \) is true.

The action of the group G on the objects in list is stored in the attribute DigraphGroup (7.2.10), and is used to speed up operations like DigraphDiameter (5.4.1).

for a Grape package graph
if obj is a Grape package graph (i.e. a record for which the function IsGraph returns true), then this function returns a digraph isomorphic to obj.

for a binary relation
if obj is a binary relation on the points [1 .. n] for some positive integer n, then this function returns the digraph defined by obj. Specifically, this function returns a digraph which has n vertices, and which has an edge with source i and range j if and only if \([i, j]\) is a pair in the binary relation obj.

for a string naming a digraph
if obj is a non-empty string, then this function returns the digraph that has name obj. Digraphs comes with a database containing a few hundred common digraph names that can be loaded in this way. Valid names include "folkman", "diamond" and "brinkmann". If the name is commonly followed by the word "graph", then it is called without writing "graph" at the end. You can explore the available graph names using ListNamedDigraphs (3.1.13). Digraph names are case and whitespace insensitive.

Note that any undirected graphs in the database are stored as symmetric digraphs, so the resulting digraph will have twice as many edges as its undirected counterpart.
The next example illustrates the uses of the fourth and fifth variants of this constructor. The resulting digraph is a strongly regular graph, and it is actually the point graph of the van Lint-Schrijver partial geometry, [vLS81]. The algebraic description is taken from the seminal paper of Calderbank and Kantor [CK86].

```gap
gap> f := GF(3^4);
GF(3^4)
gap> gamma := First(f, x -> Order(x) = 5);
Z(3^4)^64
gap> L := Union([Zero(f)], List(Group(gamma)));
[ 0*Z(3), Z(3)^0, Z(3^4)^16, Z(3^4)^32, Z(3^4)^48, Z(3^4)^64 ]
gap> omega := Union(List(L, x -> List(Difference(L, [x]), y -> x - y)));
[ Z(3)^0, Z(3), Z(3^4)^5, Z(3^4)^7, Z(3^4)^8, Z(3^4)^13, Z(3^4)^15,
  Z(3^4)^16, Z(3^4)^21, Z(3^4)^23, Z(3^4)^24, Z(3^4)^29, Z(3^4)^31,
  Z(3^4)^32, Z(3^4)^37, Z(3^4)^39, Z(3^4)^45, Z(3^4)^47, Z(3^4)^48,
  Z(3^4)^53, Z(3^4)^55, Z(3^4)^56, Z(3^4)^61, Z(3^4)^63, Z(3^4)^64,
  Z(3^4)^69, Z(3^4)^71, Z(3^4)^72, Z(3^4)^77, Z(3^4)^79 ]
gap> adj := function(x, y)
  return x - y in omega;
end;
function( x, y ) ... end
gap> digraph := Digraph(AsList(f), adj);
<immutable digraph with 81 vertices, 2430 edges>
gap> group := Group(Z(3));;
gap> act := \*;
<Operation ">*”>
gap> digraph := Digraph(group, List(f), act, adj);
<immutable digraph with 81 vertices, 2430 edges>
```

### 3.1.8 DigraphByAdjacencyMatrix

**Returns:** A digraph.
If the optional first argument \texttt{filt} is present, then this should specify the category or representation the digraph being created will belong to. For example, if \texttt{filt} is \texttt{IsMutableDigraph} (3.1.2), then the digraph being created will be mutable, if \texttt{filt} is \texttt{IsImmutableDigraph} (3.1.3), then the digraph will be immutable. If the optional first argument \texttt{filt} is not present, then \texttt{IsImmutableDigraph} (3.1.3) is used by default.

If \texttt{list} is the adjacency matrix of a digraph in the sense of \texttt{AdjacencyMatrix} (5.2.1), then this operation returns the digraph which is defined by \texttt{list}.

Alternatively, if \texttt{list} is a square boolean matrix, then this operation returns the digraph with \texttt{Length(list)} vertices which has the edge \([i,j]\) if and only if \texttt{list[i][j]} is \texttt{true}.

\begin{verbatim}
Example

 gap> DigraphByAdjacencyMatrix([  
 > [0, 1, 0, 2, 0],  
 > [1, 1, 1, 0, 1],  
 > [0, 3, 2, 1, 1],  
 > [0, 0, 1, 0, 1],  
 > [2, 0, 0, 0, 0]]);  
<immutable multidigraph with 5 vertices, 18 edges>
 gap> D := DigraphByAdjacencyMatrix([  
 > [true, false, true],  
 > [false, false, true],  
 > [false, true, false]]);  
<immutable digraph with 3 vertices, 4 edges>
 gap> OutNeighbours(D);  
[ [ 1, 3 ], [ 3 ], [ 2 ] ]
 gap> D := DigraphByAdjacencyMatrix(IsMutableDigraph,  
 > [[true, false, true],  
 > [false, false, true],  
 > [false, true, false]]);  
<mutable digraph with 3 vertices, 4 edges>
\end{verbatim}

\subsection{DigraphByEdges}

\texttt{DigraphByEdges([filt, ]list[, n])}

\textbf{Returns:} A digraph.

If the optional first argument \texttt{filt} is present, then this should specify the category or representation the digraph being created will belong to. For example, if \texttt{filt} is \texttt{IsMutableDigraph} (3.1.2), then the digraph being created will be mutable, if \texttt{filt} is \texttt{IsImmutableDigraph} (3.1.3), then the digraph will be immutable. If the optional first argument \texttt{filt} is not present, then \texttt{IsImmutableDigraph} (3.1.3) is used by default.

If \texttt{list} is list of pairs of positive integers, then this function returns the digraph with the minimum number of vertices \(m\) such that its list equal \texttt{list}.

If \texttt{list} is of list of pairs of positive integers, then this function returns the digraph with \texttt{n} vertices and list \texttt{list}.

See \texttt{DigraphEdges} (5.1.3).

\begin{verbatim}
Example

 gap> DigraphByEdges(  
 > [[1, 3 ], [2, 1 ], [2, 3 ], [2, 5 ], [3, 6 ],  
 > [4, 6 ], [5, 2 ], [5, 4 ], [5, 6 ], [6, 6]]);  
<immutable digraph with 6 vertices, 10 edges>
\end{verbatim}
Digraphs

3.1.10  EdgeOrbitsDigraph
▷ EdgeOrbitsDigraph(G, edges[, n])
      (operation)

Returns:  An immutable digraph.

If G is a permutation group, edges is an edge or list of edges, and n is a non-negative integer such that G fixes \([1 \ldots n]\) setwise, then this operation returns an immutable digraph with n vertices and the union of the orbits of the edges in edges under the action of the permutation group G. An edge in this context is simply a pair of positive integers.

If the optional third argument n is not present, then the largest moved point of the permutation group G is used by default.

Example

```
gap> digraph := EdgeOrbitsDigraph(Group((1, 3), (1, 2)(3, 4)),
    [[1, 2], [4, 5]], 5);
<immutable digraph with 5 vertices, 12 edges>
gap> OutNeighbours(digraph);
[ [ 2, 4, 5 ], [ 1, 3, 5 ], [ 2, 4, 5 ], [ 1, 3, 5 ], [ ] ]
gap> RepresentativeOutNeighbours(digraph);
[ [ 2, 4, 5 ], [ ] ]
```

3.1.11  DigraphByInNeighbours
▷ DigraphByInNeighbours([filt, ]list)
      (operation)

Returns:  A digraph.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

If list is a list of lists of positive integers list the range \([1 \ldots \text{Length(list)}]\), then this function returns the digraph with vertices \(E^0 = [1 \ldots \text{Length(list)}]\), and edges corresponding to the entries of list. More precisely, there is an edge with source vertex \(i\) and range vertex \(j\) if \(i\) is in the list list[j].

If \(i\) occurs in the list list[j] with multiplicity \(k\), then there are \(k\) multiple edges from \(i\) to \(j\). See InNeighbours (5.2.7).

Example

```
gap> D := DigraphByInNeighbours([ [2, 5, 8, 10], [2, 3, 4, 5, 6, 8, 9, 10],
    [1], [3, 5, 7, 8, 10], [2, 5, 7], [3, 6, 7, 9, 10], [1, 4],
```
Digraphs

3.1.12  CayleyDigraph

▷ CayleyDigraph(G[, gens])

Returns:  An immutable digraph.

Let \( G \) be any group and let \( \text{gens} \) be a list of elements of \( G \). This operation returns an immutable digraph that corresponds to the Cayley graph of \( G \) with respect to \( \text{gens} \).

The vertices of the digraph correspond to the elements of \( G \), in the order given by \( \text{AsList}(G) \). There exists an edge from vertex \( u \) to vertex \( v \) if and only if there exists a generator \( g \) in \( \text{gens} \) such that \( \text{AsList}(G)[u] \cdot g = \text{AsList}(G)[v] \).

The labels of the vertices \( u \), \( v \), and the edge \([u, v]\) are the corresponding elements \( \text{AsList}(G)[u] \), \( \text{AsList}(G)[v] \), and generator \( g \), respectively; see \text{DigraphVertexLabel} (5.1.9) and \text{DigraphEdgeLabel} (5.1.11).

If the optional second argument \( \text{gens} \) is not present, then the generators of \( G \) are used by default.

The digraph created by this operation belongs to the category \text{IsCayleyDigraph} (3.1.4), the group \( G \) can be recovered from the digraph using \text{GroupOfCayleyDigraph} (5.5.1), and the generators \( \text{gens} \) can be obtained using \text{GeneratorsOfCayleyDigraph} (5.5.2).

Note that this function can only return an immutable digraph.

Example

```gap
gap> G := DihedralGroup(8);
<pc group of size 8 with 3 generators>
gap> CayleyDigraph(G);
<immutable digraph with 8 vertices, 24 edges>
gap> G := DihedralGroup(IsPermGroup, 8);
Group([ [1,2,3,4], (2,4) ])
gap> CayleyDigraph(G);
<immutable digraph with 8 vertices, 16 edges>
gap> digraph := CayleyDigraph(G, [()]);
<immutable digraph with 8 vertices, 8 edges>
gap> GroupOfCayleyDigraph(digraph) = G;
true
gap> GeneratorsOfCayleyDigraph(digraph);
[ () ]
```

3.1.13  ListNamedDigraphs

▷ ListNamedDigraphs(s[, level])

Returns:  A list of strings representing digraph names.

This function searches through the list of names that are currently in the Digraphs database of named digraphs. The first argument \( s \) should be a partially completed string; this function returns all completions \( \text{str} \) of the string \( s \) such that \( \text{Digraph}(\text{str}) \) will successfully return a digraph.
The optional second argument `level` controls the flexibility of the search. If `level = 1`, then only strings beginning exactly with `s` are returned. If `level = 2`, then all names containing `s` as a substring are returned. If `level = 3`, then once again a substring search is carried out, but characters that are not alphanumeric are ignored in the search.

If `level` is not specified, it is set by default to equal 2.

The search is always case and whitespace insensitive, and this is also the case when applying Digraph (3.1.7) to a string.

### 3.2 Changing representations

#### 3.2.1 AsBinaryRelation

▷ AsBinaryRelation(digraph)

**Returns:** A binary relation.

If `digraph` is a digraph with a positive number of vertices `n`, and no multiple edges, then this operation returns a binary relation on the points `[1..n]`. The pair `[i,j]` is in the binary relation if and only if `[i,j]` is an edge in `digraph`.

**Example**

```gap
gap> D := Digraph([[3, 2], [1, 2], [2], [3, 4]]);
<immutable digraph with 4 vertices, 7 edges>
gap> AsBinaryRelation(D);
Binary Relation on 4 points
```

#### 3.2.2 AsDigraph (for a binary relation)

▷ AsDigraph([filt, ]f[, n])

**Returns:** A digraph, or fail.

If `f` is a binary relation represented as one of the following in GAP:

- a transformation satisfying `IsTransformation` (Reference: `IsTransformation`);
- a permutation satisfying `IsPerm` (Reference: `IsPerm`);
- a partial perm satisfying `IsPartialPerm` (Reference: `IsPartialPerm`);
- a binary relation satisfying `IsBinaryRelation` (Reference: `IsBinaryRelation`);

and `n` is a non-negative integer, then AsDigraph attempts to construct a digraph with `n` vertices whose edges are determined by `f`.

The digraph returned by AsDigraph has for each vertex `v` in `[1 .. n]`, an edge with source `v` and range `v ^ f`. If `v ^ f` is greater than `n` for any `v`, then `fail` is returned.

If the optional second argument `n` is not supplied, then the degree of the transformation `f`, the largest moved point of the permutation `f`, the maximum of the degree and the codegree of the partial perm `f`, or as applicable, is used by default.
If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

\begin{verbatim}
gap> f := Transformation([4, 3, 3, 1, 7, 9, 10, 4, 2, 3]);
Transformation( [ 4, 3, 3, 1, 7, 9, 10, 4, 2, 3 ] )
gap> AsDigraph(f);
<immutable functional digraph with 10 vertices>
gap> AsDigraph(f, 4);
<immutable functional digraph with 4 vertices>
gap> AsDigraph(f, 5);
fail
\end{verbatim}

3.2.3 Graph

\textbf{Graph(digraph)}

\textit{Returns:} A Grape package graph.

If \texttt{digraph} is a mutable or immutable digraph without multiple edges, then this operation returns a Grape package graph that is isomorphic to \texttt{digraph}.

If \texttt{digraph} is a multidigraph, then since Grape does not support multiple edges, the multiple edges will be reduced to a single edge in the result. In other words, for a multidigraph this operation will return the same as \texttt{Graph(DigraphRemoveAllMultipleEdges(digraph))}.

Example

\begin{verbatim}
gap> Petersen := Graph(SymmetricGroup(5), [[1, 2]], OnSets, 
> function(x, y) return Intersection(x, y) = [] ; end);
gap> Display(Petersen);
rec(
  adjacencies := [ [ 3, 5, 8 ] ],
  group := 
    Group( [ ( 1, 2, 3, 5, 7)( 4, 6, 8, 9,10), ( 2, 4)( 6, 9)( 7,10) ] ),
  isGraph := true,
  names := [ [ 1, 2 ], [ 2, 3 ], [ 3, 4 ], [ 1, 3 ], [ 4, 5 ],
            [ 2, 4 ], [ 1, 5 ], [ 3, 5 ], [ 1, 4 ], [ 2, 5 ] ],
  order := 10,
  representatives := [ 1 ],
  schreierVector := [ -1, 1, 1, 2, 1, 1, 1, 1, 2, 2 ] )
gap> Digraph(Petersen);
\end{verbatim}
3.2.4 AsGraph

▷ AsGraph(digraph) (attribute)

Returns: A Grape package graph.

If digraph is a digraph, then this method returns the same as Graph (3.2.3), except that if digraph is immutable, then the result will be stored as a mutable attribute of digraph. In this latter case, when AsGraph(digraph) is called subsequently, the same GAP object will be returned as before.

Example

```gap
gap> D := Digraph([[1, 2], [3], []]);
<immutable digraph with 3 vertices, 3 edges>
gap> G := AsGraph(D);
rec( adjacencies := [ [ 1, 2 ], [ 3 ], [ ] ], group := Group(()),
    isGraph := true, names := [ 1 .. 3 ], order := 3,
    representatives := [ 1, 2, 3 ], schreierVector := [ -1, -2, -3 ] )
```

3.2.5 AsTransformation

▷ AsTransformation(digraph) (attribute)

Returns: A transformation, or fail

If digraph is a functional digraph, then AsTransformation returns the transformation which is defined by digraph. See IsFunctionalDigraph (6.2.9). Otherwise, AsTransformation(digraph) returns fail.

If digraph is a functional digraph with n vertices, then AsTransformation(digraph) will return the transformation f of degree at most n where for each 1 ≤ i ≤ n, i f is equal to the unique out-neighbour of vertex i in digraph.

Example

```gap
gap> D := Digraph([[1, 2], [3], [2]]);
<immutable digraph with 3 vertices, 3 edges>
gap> AsTransformation(D);
Transformation( [ 1, 3, 2 ] )
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> AsTransformation(D);
Transformation( [ 2, 3, 1 ] )
gap> AsPermutation(last);
(1,2,3)
gap> D := Digraph([[2, 3], [], []]);
<immutable digraph with 3 vertices, 2 edges>
gap> AsTransformation(D);
fail
```
3.3 New digraphs from old

3.3.1 DigraphImmutableCopy

Returns: A digraph.

Each of these operations returns a new copy of `digraph`, of the appropriate mutability, retaining none of the attributes or properties of `digraph`.

DigraphCopy is a synonym for DigraphCopySameMutability.

Example

```gap
gap> D := CycleDigraph(10);
<immutable cycle digraph with 10 vertices>
gap> DigraphCopy(D) = D;
true
gap> IsIdenticalObj(DigraphCopy(D), D);
false
gap> DigraphMutableCopy(D);
<mutable digraph with 10 vertices, 10 edges>
```

3.3.2 DigraphImmutableCopyIfImmutable

Returns: A digraph.

Each of these operations returns either the original argument `digraph`, or a new copy of `digraph` of the appropriate mutability, according to the mutability of `digraph`.

Example

```gap
gap> C := CycleDigraph(10);
<immutable cycle digraph with 10 vertices>
gap> D := DigraphImmutableCopyIfImmutable(C);
<immutable digraph with 10 vertices, 10 edges>
gap> IsIdenticalObj(C, D);
false
gap> C = D;
true
gap> D := DigraphImmutableCopyIfMutable(C);
<immutable cycle digraph with 10 vertices>
gap> IsIdenticalObj(C, D);
true
gap> C = D;
true
gap> D := DigraphMutableCopyIfMutable(C);
<immutable cycle digraph with 10 vertices>
gap> IsMutableDigraph(D);
false
```
Digraphs

3.3.3 InducedSubdigraph

▷ InducedSubdigraph(digraph, verts)

(operation)

Returns: A digraph.

If digraph is a digraph, and verts is a subset of the vertices of digraph, then this operation returns a digraph constructed from digraph by retaining precisely those vertices in verts, and those edges whose source and range vertices are both contained in verts.

The vertices of the induced subdigraph are \([1..\text{Length}(verts)]\) but the original vertex labels can be accessed via DigraphVertexLabels (5.1.10).

If digraph belongs to IsMutableDigraph (3.1.2), then digraph is modified in place. If digraph belongs to IsImmutableDigraph (3.1.3), a new immutable digraph containing the appropriate vertices and edges is returned.

```
gap> D := DigraphMutableCopyIfImmutable(C);
<mutable digraph with 10 vertices, 10 edges>
gap> IsMutableDigraph(D);
true
gap> C := CycleDigraph(IsMutableDigraph, 10);
<mutable digraph with 10 vertices, 10 edges>
gap> D := DigraphImmutableCopyIfImmutable(C);
<mutable digraph with 10 vertices, 10 edges>
gap> IsIdenticalObj(C, D);
true
gap> C = D;
true
gap> D := DigraphImmutableCopyIfMutable(C);
<immutable digraph with 10 vertices, 10 edges>
gap> IsIdenticalObj(C, D);
false
gap> C = D;
true
gap> D := DigraphMutableCopyIfMutable(C);
<mutable digraph with 10 vertices, 10 edges>
gap> IsMutableDigraph(D);
true
gap> D := DigraphMutableCopyIfImmutable(C);
<mutable digraph with 10 vertices, 10 edges>
gap> IsIdenticalObj(C, D);
true
gap> IsMutableDigraph(D);
true
```

```
3.3.3 InducedSubdigraph

▷ InducedSubdigraph(digraph, verts)

(operation)

Returns: A digraph.

If digraph is a digraph, and verts is a subset of the vertices of digraph, then this operation returns a digraph constructed from digraph by retaining precisely those vertices in verts, and those edges whose source and range vertices are both contained in verts.

The vertices of the induced subdigraph are \([1..\text{Length}(verts)]\) but the original vertex labels can be accessed via DigraphVertexLabels (5.1.10).

If digraph belongs to IsMutableDigraph (3.1.2), then digraph is modified in place. If digraph belongs to IsImmutableDigraph (3.1.3), a new immutable digraph containing the appropriate vertices and edges is returned.

```
Example

```
```
3.3.4 ReducedDigraph

Returns: A digraph.

This function returns a digraph isomorphic to the subdigraph of `digraph` induced by the set of non-isolated vertices, i.e. the set of those vertices of `digraph` which are the source or range of some edge in `digraph`. See InducedSubdigraph (3.3.3).

The ordering of the remaining vertices of `digraph` is preserved, as are the labels of the remaining vertices and edges; see DigraphVertexLabels (5.1.10) and DigraphEdgeLabels (5.1.12). This can allow one to match a vertex in the reduced digraph to the corresponding vertex in `digraph`.

If `digraph` is immutable, then a new immutable digraph is returned. Otherwise, the isolated vertices of the mutable digraph `digraph` are removed in-place.

```
<mutable multidigraph with 3 vertices, 9 edges>
gap> D := new;
true
```

Example

```
gap> D := Digraph([[1, 2], [], [], [1, 4], []]);
<immutable digraph with 5 vertices, 4 edges>
gap> R := ReducedDigraph(D);
<immutable digraph with 3 vertices, 4 edges>
gap> OutNeighbours(R);
[ [ 1, 2 ], [ ], [ 1, 3 ] ]
gap> DigraphEdges(D);
[ [ 1, 1 ], [ 1, 2 ], [ 4, 1 ], [ 4, 4 ] ]
gap> DigraphEdges(R);
[ [ 1, 1 ], [ 1, 2 ], [ 3, 1 ], [ 3, 3 ] ]
gap> DigraphVertexLabel(R, 3);
4
```

3.3.5 MaximalSymmetricSubdigraph

Returns: A digraph.

If `digraph` is a digraph, then MaximalSymmetricSubdigraph returns a symmetric digraph without multiple edges which has the same vertex set as `digraph`, and whose edge list is formed from
Digraphs

A digraph is a directed graph where edges have orientations. The digraph returned by MaximalSymmetricSubdigraphWithoutLoops is the same, except that loops are removed.

If digraph is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph digraph is changed in-place into such a digraph described above.

See IsSymmetricDigraph (6.2.14), IsMultiDigraph (6.2.11), and DigraphHasLoops (6.2.1) for more information.

Example

```
gap> D := Digraph([[2, 2], [1, 3], [4], [3, 1]]);  
<immutable multidigraph with 4 vertices, 7 edges>
gap> not IsSymmetricDigraph(D) and IsMultiDigraph(D);  
true  
gap> OutNeighbours(D);  
[ [ 2, 2 ], [ 1, 3 ], [ 4 ], [ 3, 1 ] ]  
gap> S := MaximalSymmetricSubdigraph(D);  
<immutable symmetric digraph with 4 vertices, 4 edges>
gap> IsSymmetricDigraph(S) and not IsMultiDigraph(S);  
true  
gap> OutNeighbours(S);  
[ [ 2 ], [ 1 ], [ 4 ], [ 3 ] ]  
gap> D := CycleDigraph(IsMutableDigraph, 3);  
<mutable digraph with 3 vertices, 3 edges>
gap> MaximalSymmetricSubdigraph(D);  
<mutable empty digraph with 3 vertices>
gap> D;  
<mutable empty digraph with 3 vertices>
```

### 3.3.6 MaximalAntiSymmetricSubdigraph

- **MaximalAntiSymmetricSubdigraph(digraph)** (operation)
- **MaximalAntiSymmetricSubdigraphAttr(digraph)** (attribute)

**Returns:** A digraph.

If digraph is a digraph, then MaximalAntiSymmetricSubdigraph returns an anti-symmetric subdigraph of digraph formed by retaining the vertices of digraph, discarding any duplicate edges, and discarding any edge \([i,j]\) of digraph where \(i > j\) and the reverse edge \([j,i]\) is an edge of digraph. In other words, for every symmetric pair of edges \([i,j]\) and \([j,i]\) in digraph, where \(i\) and \(j\) are distinct, it discards the edge \([\text{max}(i,j),\text{min}(i,j)]\).

If digraph is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph digraph is changed in-place.

See IsAntisymmetricDigraph (6.2.2) for more information.

Example

```
gap> D := Digraph([[2, 2], [1, 3], [4], [3, 1]]);  
<immutable multidigraph with 4 vertices, 7 edges>
gap> not IsAntiSymmetricDigraph(D) and IsMultiDigraph(D);  
true  
gap> OutAntiSymmetricDigraph(D);  
[ [ 2, 2 ], [ 1, 3 ], [ 4 ], [ 3, 1 ] ]  
gap> D := MaximalAntiSymmetricSubdigraph(D);  
<immutable antisymmetric digraph with 4 vertices, 4 edges>
```
Digraphs

gap> IsAntiSymmetricDigraph(D) and not IsMultiDigraph(D);
true
gap> OutNeighbours(D);
[ [ 2 ], [ 3 ], [ 4 ], [ 1 ] ]
gap> D := Digraph(IsMutableDigraph, [[2], [1]]);
<mutable digraph with 2 vertices, 2 edges>
gap> MaximalAntiSymmetricSubdigraph(D);
<mutable digraph with 2 vertices, 1 edge>
gap> D;
<mutable digraph with 2 vertices, 1 edge>

3.3.7 UndirectedSpanningForest

▷ UndirectedSpanningForest(digraph) (operation)
▷ UndirectedSpanningForestAttr(digraph) (attribute)
▷ UndirectedSpanningTree(digraph) (operation)
▷ UndirectedSpanningTreeAttr(digraph) (attribute)

Returns: A digraph, or fail.

If digraph is a digraph with at least one vertex, then UndirectedSpanningForest returns an undirected spanning forest of digraph, otherwise this attribute returns fail. See IsUndirectedSpanningForest (4.1.2) for the definition of an undirected spanning forest.

If digraph is a digraph with at least one vertex and whose MaximalSymmetricSubdigraph (3.3.5) is connected (see IsConnectedDigraph (6.6.3)), then UndirectedSpanningTree returns an undirected spanning tree of digraph, otherwise this attribute returns fail. See IsUndirectedSpanningTree (4.1.2) for the definition of an undirected spanning tree.

If digraph is immutable, then an immutable digraph is returned. Otherwise, the mutable digraph digraph is changed in-place into an undirected spanning tree of digraph.

Note that for an immutable digraph that has known undirected spanning tree, the attribute UndirectedSpanningTree returns the same digraph as the attribute UndirectedSpanningForest.

Example

gap> D := Digraph([[1, 2, 1, 3], [1], [4], [3, 4, 3]]);
<immutable multidigraph with 4 vertices, 9 edges>
gap> UndirectedSpanningTree(D);
fail
gap> forest := UndirectedSpanningForest(D);
<immutable undirected forest digraph with 4 vertices, 4 edges>
gap> OutNeighbours(forest);
[ [ 2 ], [ 1 ], [ 4 ], [ 3 ] ]
gap> IsUndirectedSpanningForest(D, forest);
true
gap> DigraphConnectedComponents(forest).comps;
[ [ 1, 2 ], [ 3, 4 ] ]
gap> DigraphConnectedComponents(MaximalSymmetricSubdigraph(D)).comps;
[ [ 1, 2 ], [ 3, 4 ] ]
gap> UndirectedSpanningForest(MaximalSymmetricSubdigraph(D)) > = forest;
true
gap> D := CompleteDigraph(4);
<immutable complete digraph with 4 vertices>
gap> tree := UndirectedSpanningTree(D);
3.3.8 DigraphShortestPathSpanningTree

▷ DigraphShortestPathSpanningTree(digraph, v)  
(operation)

Returns: A digraph, or fail.

If \( v \) is a vertex in \( \text{digraph} \) and every other vertex of \( \text{digraph} \) is reachable from \( v \), then this operation returns the shortest path spanning tree of \( \text{digraph} \) rooted at \( v \). If there exist vertices in \( \text{digraph} \) (other than \( v \)) that are not reachable from \( v \), then \( \text{DigraphShortestPathSpanningTree} \) returns fail. See \( \text{IsReachable} \) (5.4.19).

The shortest path spanning tree of \( \text{digraph} \) rooted at \( v \) is a subdigraph of \( \text{digraph} \) that is a directed tree, with unique source vertex \( v \), and where for each other vertex \( u \) in \( \text{digraph} \), the unique directed path from \( v \) to \( u \) in the tree is the lexicographically-least shortest directed path from \( v \) to \( u \) in \( \text{digraph} \).

See \( \text{IsDirectedTree} \) (6.6.8), \( \text{DigraphSources} \) (5.1.7), and \( \text{DigraphShortestPath} \) (5.4.23).

If \( \text{digraph} \) belongs to \( \text{IsImmutableDigraph} \) (3.1.3), then \( \text{digraph} \) is modified in place. If \( \text{digraph} \) belongs to \( \text{IsMutableDigraph} \) (3.1.2), then \( \text{digraph} \) is modified in place. If \( \text{digraph} \) belongs to \( \text{IsImmutableDigraph} \) (3.1.3), then the spanning tree is a new immutable digraph.

Example

\[ \text{} \]
\[ \text{gap} \triangleright D := \text{Digraph}([ [1, 2], [3], [2, 4], [1], [2, 4] ]); \]
\[ \text{<immutable digraph with 5 vertices, 8 edges>} \]
\[ \text{gap} \triangleright \text{ForAll}([2 .. 5], v -> \text{IsReachable}(D, 1, v)); \]
\[ \text{false} \]
\[ \text{gap} \triangleright \text{DigraphShortestPathSpanningTree}(D, 1); \]
\[ \text{fail} \]
\[ \text{gap} \triangleright \text{tree} := \text{DigraphShortestPathSpanningTree}(D, 5); \]
\[ \text{<immutable directed tree digraph with 5 vertices>} \]
\[ \text{gap} \triangleright \text{OutNeighbours}(\text{tree}); \]
\[ \text{gap} \triangleright \text{ForAll}(\text{DigraphVertices}(D), v -> \]
\[ > \text{DigraphShortestPath}(D, 5, v) = \text{DigraphPath}(\text{tree}, 5, v)); \]
\[ \text{true} \]

3.3.9 QuotientDigraph

▷ QuotientDigraph(digraph, p)  
(operation)

Returns: A digraph.
If \textit{digraph} is a digraph, and \textit{p} is a partition of the vertices of \textit{digraph}, then this operation returns a digraph constructed by amalgamating all vertices of \textit{digraph} which lie in the same part of \textit{p}.

A partition of the vertices of \textit{digraph} is a list of non-empty disjoint lists, such that the union of all the sub-lists is equal to vertex set of \textit{digraph}. In particular, each vertex must appear in precisely one sub-list.

The vertices of \textit{digraph} in part \textit{i} of \textit{p} will become vertex \textit{i} in the quotient, and there exists some edge in \textit{digraph} with source in part \textit{i} and range in part \textit{j} if and only if there is an edge from \textit{i} to \textit{j} in the quotient. In particular, this means that the quotient of a digraph has no multiple edges. which was a change introduced in version 1.0.0 of the \texttt{Digraphs} package.

If \textit{digraph} belongs to \texttt{IsMutableDigraph} (3.1.2), then \textit{digraph} is modified in place. If \textit{digraph} belongs to \texttt{IsImmutableDigraph} (3.1.3), a new immutable digraph with the above properties is returned.

\begin{verbatim}
Example
gap> D := Digraph([[2, 1], [4], [1], [1, 3, 4]]);
<immutable digraph with 4 vertices, 7 edges>
gap> DigraphVertices(D);
[ 1 .. 4 ]
gap> DigraphEdges(D);
[ [ 1, 2 ], [ 1, 1 ], [ 2, 4 ], [ 3, 1 ], [ 4, 1 ], [ 4, 3 ],
  [ 4, 4 ] ]
gap> p := [[1], [2, 4], [3]];
[ [ 1 ], [ 2, 4 ], [ 3 ] ]
gap> quo := QuotientDigraph(D, p);
<immutable digraph with 3 vertices, 6 edges>
gap> DigraphVertices(quo);
[ 1 .. 3 ]
gap> DigraphEdges(quo);
[ [ 1, 1 ], [ 1, 2 ], [ 2, 1 ], [ 2, 2 ], [ 2, 3 ], [ 3, 1 ] ]
gap> QuotientDigraph(EmptyDigraph(0), []);
<immutable empty digraph with 0 vertices>
\end{verbatim}

\subsection{DigraphReverse}

\begin{itemize}
\item \texttt{DigraphReverse(digraph)} (operation)
\item \texttt{DigraphReverseAttr(digraph)} (attribute)
\end{itemize}

\textbf{Returns:} A digraph.

The reverse of a digraph is the digraph formed by reversing the orientation of each of its edges, i.e. for every edge \([i, j]\) of a digraph, the reverse contains the corresponding edge \([j, i]\).

\texttt{DigraphReverse} returns the reverse of the digraph \textit{digraph}. If \textit{digraph} is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph \textit{digraph} is changed in-place into its reverse.

\begin{verbatim}
Example
gap> D := Digraph([[3], [1, 3, 5], [1], [1, 2, 4], [2, 3, 5]]);
<immutable digraph with 5 vertices, 11 edges>
gap> DigraphReverse(D);
<immutable digraph with 5 vertices, 11 edges>
gap> OutNeighbours(last);
[ [ 2, 3, 4 ], [ 4, 5 ], [ 1, 2, 5 ], [ 4 ], [ 2, 5 ] ]
gap> D := Digraph([[2, 4], [1], [4], [3, 4]]);
<immutable digraph with 4 vertices, 6 edges>
\end{verbatim}
3.3.11 DigraphDual

\[ \text{DigraphDual(digraph)} \] (operation)
\[ \text{DigraphDualAttr(digraph)} \] (attribute)

Returns: A digraph.

The dual of digraph has the same vertices as digraph, and there is an edge in the dual from \( i \) to \( j \) whenever there is no edge from \( i \) to \( j \) in digraph. The dual is sometimes called the complement.

DigraphDual returns the dual of the digraph digraph. If digraph is an immutable digraph, then a new immutable digraph is returned. Otherwise, the mutable digraph digraph is changed in-place into its dual.

Example

\[ \text{gap> D := Digraph([ [2, 3], [4, 6], [5], [7, 8, 9], [], [], [], [] ]);} \]
\[ \text{<immutable digraph with 9 vertices, 8 edges>} \]
\[ \text{gap> DigraphDual(D);} \]
\[ \text{<immutable digraph with 9 vertices, 73 edges>} \]

3.3.12 DigraphSymmetricClosure

\[ \text{DigraphSymmetricClosure(digraph)} \] (operation)
\[ \text{DigraphSymmetricClosureAttr(digraph)} \] (attribute)

Returns: A digraph.

If digraph is a digraph, then this attribute gives the minimal symmetric digraph which has the same vertices and contains all the edges of digraph.

A digraph is symmetric if its adjacency matrix AdjacencyMatrix (5.2.1) is symmetric. For a digraph with multiple edges this means that there are the same number of edges from a vertex \( u \) to a vertex \( v \) as there are from \( v \) to \( u \); see IsSymmetricDigraph (6.2.14).

If digraph is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph digraph is changed in-place into its symmetric closure.
Example

```gap
<# immutable digraph with 4 vertices, 8 edges #>
D := Digraph([ [ 1, 2, 3 ], [ 2, 4 ], [ 1 ], [ 3, 4 ] ]);  
<# immutable symmetric digraph with 4 vertices, 11 edges #>
D := DigraphSymmetricClosure(D);  
<# IsSymmetricDigraph(D); #>
true  
<# List(OutNeighbours(D), AsSet); #>
[ [ 1, 2, 3 ], [ 1, 2, 4 ], [ 1, 4 ], [ 2, 3, 4 ] ]  
<# immutable multidigraph with 2 vertices, 3 edges #>
D := Digraph([ [ 2, 2 ], [ 1 ] ]);  
<# immutable symmetric multidigraph with 2 vertices, 4 edges #>
D := DigraphSymmetricClosure(D);  
<# OutNeighbours(D); #>
[ [ 2, 2 ], [ 1, 1 ] ]  
<# CycleDigraph(IsMutableDigraph, 3); #>
<# DigraphSymmetricClosure(D); #>
<# D; #>
<# mutable digraph with 3 vertices, 6 edges #>
D := CycleDigraph(IsMutableDigraph, 3);  
D := DigraphSymmetricClosure(D);  
D;  
<# mutable digraph with 3 vertices, 6 edges #>
```

### 3.3.13 DigraphTransitiveClosure

<table>
<thead>
<tr>
<th>Operation/Attribute</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>DigraphTransitiveClosure(digraph)</td>
<td>(operation)</td>
</tr>
<tr>
<td>DigraphReflexiveTransitiveClosure(digraph)</td>
<td>(operation)</td>
</tr>
<tr>
<td>DigraphReflexiveTransitiveClosureAttr(digraph)</td>
<td>(attribute)</td>
</tr>
<tr>
<td>DigraphTransitiveClosureAttr(digraph)</td>
<td>(attribute)</td>
</tr>
</tbody>
</table>

**Returns:** A digraph.

If `digraph` is a digraph with no multiple edges, then these attributes return the (reflexive) transitive closure of `digraph`.

A digraph is reflexive if it has a loop at every vertex, and it is transitive if whenever \([i,j]\) and \([j,k]\) are edges of `digraph`, \([i,k]\) is also an edge. The (reflexive) transitive closure of a digraph `digraph` is the least (reflexive and) transitive digraph containing `digraph`.

If `digraph` is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph `digraph` is changed in-place into its (reflexive) transitive closure.

Let \(n\) be the number of vertices of `digraph`, and let \(m\) be the number of edges. For an arbitrary digraph, these attributes will use a version of the Floyd-Warshall algorithm, with complexity \(O(n^3)\). However, for a topologically sortable digraph [see DigraphTopologicalSort (5.1.8)], these attributes will use methods with complexity \(O(m + n + m \cdot n)\) when this is faster.

Example

```gap
<# immutable digraph with 9 vertices, 8 edges #>
D := DigraphFromDiSparse6String(".H’eOWR’Ul^"));  
<# immutable digraph with 9 vertices, 8 edges #>
<# IsReflexiveDigraph(D) or IsTransitiveDigraph(D); #>
false  
<# OutNeighbours(D); #>
[ [ 4, 6 ], [ 1, 3 ], [ ], [ 5 ], [ ], [ 7, 8, 9 ], [ ], [ ], [ ] ]  
<# immutable transitive digraph with 9 vertices, 18 edges #>
T := DigraphTransitiveClosure(D);  
<# immutable transitive digraph with 9 vertices, 18 edges #>
```
3.3.14 DigraphTransitiveReduction

▷ DigraphTransitiveReduction(digraph) (operation)
▷ DigraphTransitiveReductionAttr(digraph) (attribute)
▷ DigraphReflexiveTransitiveReduction(digraph) (operation)
▷ DigraphReflexiveTransitiveReductionAttr(digraph) (attribute)

Returns: A digraph.

If digraph is a topologically sortable digraph [see DigraphTopologicalSort (5.1.8)] with no multiple edges, then these operations return the (reflexive) transitive reduction of digraph.

The (reflexive) transitive reduction of such a digraph is the unique least subgraph such that the (reflexive) transitive closure of the subgraph is equal to the (reflexive) transitive closure of digraph [see DigraphReflexiveTransitiveClosure (3.3.13)]. In order words, it is the least subgraph of digraph which retains the same reachability as digraph.

If digraph is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph digraph is changed in-place into its (reflexive) transitive reduction.

Let \( n \) be the number of vertices of an arbitrary digraph, and let \( m \) be the number of edges. Then these operations use methods with complexity \( O(m + n + m \cdot n) \).

Example

gap> D := Digraph([[1, 2, 3], [3], [3]]);;
gap> DigraphHasLoops(D);
true
gap> D1 := DigraphReflexiveTransitiveReduction(D);
<immutable digraph with 3 vertices, 2 edges>
gap> DigraphHasLoops(D1);
false
gap> OutNeighbours(D1);
[ [ 2 ], [ 3 ], [  ] ]
gap> D2 := DigraphTransitiveReduction(D);
<immutable digraph with 3 vertices, 4 edges>
gap> DigraphHasLoops(D2);
true
gap> OutNeighbours(D2);
[ [ 2, 1 ], [ 3 ], [ 3 ] ]
gap> DigraphReflexiveTransitiveClosure(D)
> = DigraphReflexiveTransitiveClosure(D1);
true
gap> DigraphTransitiveClosure(D)
> = DigraphTransitiveClosure(D2);
true
gap> D := Digraph(IsMutableDigraph, [[1], [1], [1, 2, 3]]);
<mutable digraph with 3 vertices, 5 edges>
gap> DigraphReflexiveTransitiveReduction(D);
<mutable digraph with 3 vertices, 2 edges>
gap> D;
<mutable digraph with 3 vertices, 2 edges>

### 3.3.15 DigraphAddVertex

*DigraphAddVertex(digraph[, label]*)  
(operation)

**Returns:** A digraph.

The operation returns a digraph constructed from `digraph` by adding a single new vertex, and no new edges.

If the optional second argument `label` is a GAP object, then the new vertex will be labelled `label`.

If `digraph` belongs to `IsMutableDigraph` (3.1.2), then the vertex is added directly to `digraph`. If `digraph` belongs to `IsImmutableDigraph` (3.1.3), an immutable copy of `digraph` with the additional vertex is returned.

```gap
gap> D := CompleteDigraph(3);
<immutable complete digraph with 3 vertices>
gap> new := DigraphAddVertex(D);
<immutable digraph with 4 vertices, 6 edges>
gap> D = new;
false
gap> DigraphVertices(new);
[ 1 .. 4 ]
gap> new := DigraphAddVertex(D, Group([[(1, 2)]]);
<immutable digraph with 4 vertices, 6 edges>
gap> DigraphVertexLabels(new);
[ 1, 2, 3, Group([ (1,2) ]) ]
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 2, 3);
<mutable digraph with 5 vertices, 12 edges>
gap> new := DigraphAddVertex(D);
<mutable digraph with 6 vertices, 12 edges>
gap> D = new;
true
```

### 3.3.16 DigraphAddVertices (for a digraph and an integer)

*DigraphAddVertices(digraph, m)*  
(operation)

*DigraphAddVertices(digraph, labels)*  
(operation)

**Returns:** A digraph.

For a non-negative integer `m`, this operation returns a digraph constructed from `digraph` by adding `m` new vertices.
Otherwise, if \texttt{labels} is a list consisting of \( k \) \texttt{GAP} objects, then this operation returns a digraph constructed from \texttt{digraph} by adding \( k \) new vertices, which are labelled according to this list.

If \texttt{digraph} belongs to \texttt{IsMutableDigraph (3.1.2)}, then the vertices are added directly to \texttt{digraph}, which is changed in-place. If \texttt{digraph} belongs to \texttt{IsImmutableDigraph (3.1.3)}, then \texttt{digraph} itself is returned if no vertices are added (i.e. \( m=0 \) or \texttt{labels} is empty), otherwise the result is a new immutable digraph.

\begin{verbatim}
Example

gap> D := CompleteDigraph(3);
<immutable complete digraph with 3 vertices>
gap> new := DigraphAddVertices(D, 3);
<immutable digraph with 6 vertices, 6 edges>
gap> DigraphVertices(new);
[ 1 .. 6 ]
gap> new := DigraphAddVertices(D, [Group([[(1,2)]]), "d"]);
<immutable digraph with 5 vertices, 6 edges>
gap> DigraphVertexLabels(new);
[ 1, 2, 3, Group([ (1,2) ]), "d" ]
gap> DigraphAddVertices(D, 0) = D;
true

Example

gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 2, 3);
<mutable digraph with 5 vertices, 12 edges>
gap> new := DigraphAddVertices(D, 4);
<mutable digraph with 9 vertices, 12 edges>
gap> D = new;
true
\end{verbatim}

### 3.3.17 DigraphAddEdge (for a digraph and an edge)

\begin{verbatim}
▷ DigraphAddEdge(digraph, edge)
▷ DigraphAddEdge(digraph, src, ran)
\end{verbatim}

\textbf{Returns:} A digraph.

If \texttt{edge} is a pair of vertices of \texttt{digraph}, or \texttt{src} and \texttt{ran} are vertices of \texttt{digraph}, then this operation returns a digraph constructed from \texttt{digraph} by adding a new edge with source \texttt{edge[1]} and range \texttt{edge[2]}.

If \texttt{digraph} belongs to \texttt{IsMutableDigraph (3.1.2)}, then the edge is added directly to \texttt{digraph}. If \texttt{digraph} belongs to \texttt{IsImmutableDigraph (3.1.3)}, then an immutable copy of \texttt{digraph} with the additional edge is returned.

\begin{verbatim}
Example

gap> D1 := Digraph([[2], [3], []]);
<immutable digraph with 3 vertices, 2 edges>
gap> DigraphEdges(D1);
[ [ 1, 2 ], [ 2, 3 ] ]
gap> D2 := DigraphAddEdge(D1, [3, 1]);
<immutable digraph with 3 vertices, 3 edges>
gap> DigraphEdges(D2);
[ [ 1, 2 ], [ 2, 3 ], [ 3, 1 ] ]
gap> D3 := DigraphAddEdge(D2, [2, 3]);
<immutable multidigraph with 3 vertices, 4 edges>

Example

gap> D1 := Digraph([[2], [3], []]);
<immutable digraph with 3 vertices, 2 edges>
gap> DigraphEdges(D1);
[ [ 1, 2 ], [ 2, 3 ] ]
gap> D2 := DigraphAddEdge(D1, [3, 1]);
<immutable digraph with 3 vertices, 3 edges>
gap> DigraphEdges(D2);
[ [ 1, 2 ], [ 2, 3 ], [ 3, 1 ] ]
\end{verbatim}
Digraphs

3.3.18 DigraphAddEdgeOrbit

▷ DigraphAddEdgeOrbit(digraph, edge)  (operation)

Returns: A new digraph.

This operation returns a new digraph with the same vertices and edges as digraph and with additional edges consisting of the orbit of the edge edge under the action of the DigraphGroup (7.2.10) of digraph. If edge is already an edge in digraph, then digraph is returned unchanged. The argument digraph must be an immutable digraph.

An edge is simply a pair of vertices of digraph.

Example

```gap
gap> gr1 := CayleyDigraph(DihedralGroup(8));
<immutable digraph with 8 vertices, 24 edges>
gap> gr2 := DigraphAddEdgeOrbit(gr1, [1, 8]);
<immutable digraph with 8 vertices, 32 edges>
gap> DigraphEdges(gr1);
[ [ 1, 2 ], [ 1, 3 ], [ 1, 4 ], [ 2, 1 ], [ 2, 5 ], [ 2, 6 ],
  [ 3, 8 ], [ 3, 4 ], [ 3, 7 ], [ 4, 6 ], [ 4, 7 ], [ 4, 1 ],
  [ 5, 7 ], [ 5, 6 ], [ 5, 8 ], [ 6, 4 ], [ 6, 8 ], [ 6, 2 ],
  [ 7, 5 ], [ 7, 1 ], [ 7, 3 ], [ 8, 3 ], [ 8, 2 ], [ 8, 5 ] ]
```  

3.3.19 DigraphAddEdges

▷ DigraphAddEdges(digraph, edges)  (operation)

Returns: A digraph.

If edges is a (possibly empty) list of pairs of vertices of digraph, then this operation returns a digraph constructed from digraph by adding the edges specified by edges. More precisely, for every edge in edges, a new edge will be added with source edge[1] and range edge[2].

If an edge is included in edges with multiplicity k, then it will be added k times. If digraph belongs to IsMutableDigraph (3.1.2), then the edges are added directly to digraph. If digraph belongs to IsImmutableDigraph (3.1.3), then the result is returned as an immutable digraph.
Example

```gap
func := function(n)
  local source, range, i;
  source := [];
  range := [];
  for i in [1 .. n - 2] do
    Add(source, i);
    Add(range, i + 1);
  od;
  return Digraph(n, source, range);
end;

D := func(1024);
<immutable digraph with 1024 vertices, 1022 edges>
new := DigraphAddEdges(D,
  [[1023, 1024], [1, 1024], [1023, 1024], [1024, 1]]);
<immutable multidigraph with 1024 vertices, 1026 edges>
D = new;
false
D2 := DigraphMutableCopy(func(1024));
<mutable digraph with 1024 vertices, 1022 edges>
new := DigraphAddEdges(D2,
  [[1023, 1024], [1, 1024], [1023, 1024], [1024, 1]]);
<mutable multidigraph with 1024 vertices, 1026 edges>
D2 = new;
true
```

### 3.3.20 DigraphRemoveVertex

▷ DigraphRemoveVertex(digraph, v)

**Returns:** A digraph.

If \( v \) is a vertex of \( \text{digraph} \), then this operation returns a digraph constructed from \( \text{digraph} \) by removing vertex \( v \) along with any edge whose source or range vertex is \( v \).

If \( \text{digraph} \) has \( n \) vertices, then the vertices of the returned digraph are \([1 \ldots n-1]\), but the original labels can be accessed via \( \text{DigraphVertexLabels} \) (5.1.10).

If \( \text{digraph} \) belongs to \( \text{IsMutableDigraph} \) (3.1.2), then the vertex is removed directly from \( \text{digraph} \). If \( \text{digraph} \) belongs to \( \text{IsImmutableDigraph} \) (3.1.3), an immutable copy of \( \text{digraph} \) without the vertex is returned.

```
D := Digraph(["a", "b", "c"],
  [["a", "a", "b", "c", "c"],
   ["b", "c", "a", "a", "c"]]);
<immutable digraph with 3 vertices, 5 edges>
DigraphVertexLabels(D);
[ "a", "b", "c" ]
DigraphEdges(D);
[ [ 1, 2 ], [ 1, 3 ], [ 2, 1 ], [ 3, 1 ], [ 3, 3 ] ]
new := DigraphRemoveVertex(D, 2);
<immutable digraph with 2 vertices, 3 edges>
DigraphVertexLabels(new);
[ "a", "c" ]
D := CycleDigraph(IsMutableDigraph, 5);
```
<mutable digraph with 5 vertices, 5 edges>
gap> new := DigraphRemoveVertex(D, 1);
<mutable digraph with 4 vertices, 3 edges>
gap> DigraphVertexLabels(D);
[ 2, 3, 4, 5 ]
gap> D = new;
true

### 3.3.21 DigraphRemoveVertices

**Operation**

```plaintext`
DigraphRemoveVertices(digraph, verts)
```

**Returns:** A digraph.

If `verts` is a (possibly empty) duplicate-free list of vertices of `digraph`, then this operation returns a digraph constructed from `digraph` by removing every vertex in `verts`, along with any edge whose source or range vertex is in `verts`.

If `digraph` has `n` vertices, then the vertices of the new digraph are `[1 .. n-Length(verts)]`, but the original labels can be accessed via `DigraphVertexLabels(5.1.10)`.

If `digraph` belongs to `IsMutableDigraph(3.1.2)`, then the vertices are removed directly from `digraph`. If `digraph` belongs to `IsImmutableDigraph(3.1.3)`, an immutable copy of `digraph` without the vertices is returned.

#### Example

```plaintext`
gap> D := Digraph([[3], [1, 3, 5], [1], [1, 2, 4], [2, 3, 5]]);
<immutable digraph with 5 vertices, 11 edges>
gap> SetDigraphVertexLabels(D, ["a", "b", "c", "d", "e"]);
gap> new := DigraphRemoveVertices(D, [2, 4]);
<immutable digraph with 3 vertices, 4 edges>
gap> DigraphVertexLabels(new);
[ "a", "c", "e" ]
gap> D := CycleDigraph(IsMutableDigraph, 5);
<mutable digraph with 5 vertices, 5 edges>
gap> new := DigraphRemoveVertices(D, [1, 3]);
<mutable digraph with 3 vertices, 1 edge>
gap> DigraphVertexLabels(D);
[ 2, 4, 5 ]
gap> D = new;
true
```

### 3.3.22 DigraphRemoveEdge (for a digraph and an edge)

**Operation**

```plaintext`
DigraphRemoveEdge(digraph, edge)
```

**Returns:** A digraph.

If `digraph` is a digraph with no multiple edges and `edge` is a pair of vertices of `digraph`, or `src` and `ran` are vertices of `digraph`, then this operation returns a digraph constructed from `digraph` by removing the edge specified by `edge` or `[src, ran]`.

If `digraph` belongs to `IsMutableDigraph(3.1.2)`, then the edge is removed directly from `digraph`. If `digraph` belongs to `IsImmutableDigraph(3.1.3)`, an immutable copy of `digraph` without the edge is returned.
Note that if `digraph` belongs to `IsImmutableDigraph (3.1.3)`, then a new copy of `digraph` will be returned even if `edge` or `[src, ran]` does not define an edge of `digraph`.

Example

```gap
gap> D := CycleDigraph(250000);;
<immutable cycle digraph with 250000 vertices>
gap> D := DigraphRemoveEdge(D, [250000, 1]);;
<immutable digraph with 250000 vertices, 249999 edges>
gap> new := DigraphRemoveEdge(D, [25000, 2]);;
gap> new = D;
true

gap> IsIdenticalObj(new, D);
false

gap> D := DigraphMutableCopy(D);;
gap> new := DigraphRemoveEdge(D, 2500, 2);;
gap> IsIdenticalObj(new, D);
true
```

### 3.3.23 DigraphRemoveEdgeOrbit

- **DigraphRemoveEdgeOrbit**(digraph, edge)  
  (operation)

**Returns:** A new digraph.

This operation returns a new digraph with the same vertices as `digraph` and with the orbit of the edge `edge` (under the action of the `DigraphGroup (7.2.10)` of `digraph`) removed. If `edge` is not an edge in `digraph`, then `digraph` is returned unchanged. The argument `digraph` must be an immutable digraph.

An edge is simply a pair of vertices of `digraph`.

Example

```gap
gap> gr1 := CayleyDigraph(DihedralGroup(8));
<immutable digraph with 8 vertices, 24 edges>
gap> gr2 := DigraphAddEdgeOrbit(gr1, [1, 8]);
<immutable digraph with 8 vertices, 32 edges>
gap> DigraphEdges(gr1);
[ [ 1, 2 ], [ 1, 3 ], [ 1, 4 ], [ 2, 1 ], [ 2, 5 ], [ 2, 6 ],
  [ 3, 8 ], [ 3, 4 ], [ 3, 7 ], [ 4, 6 ], [ 4, 7 ], [ 4, 1 ],
  [ 5, 7 ], [ 5, 6 ], [ 5, 8 ], [ 6, 4 ], [ 6, 8 ], [ 6, 2 ],
  [ 7, 5 ], [ 7, 1 ], [ 7, 3 ], [ 8, 3 ], [ 8, 2 ], [ 8, 5 ] ]
gap> DigraphEdges(gr2);
[ [ 1, 2 ], [ 1, 3 ], [ 1, 4 ], [ 1, 8 ], [ 2, 1 ], [ 2, 5 ],
  [ 2, 6 ], [ 2, 7 ], [ 3, 8 ], [ 3, 4 ], [ 3, 7 ], [ 3, 6 ],
  [ 4, 6 ], [ 4, 7 ], [ 4, 1 ], [ 4, 5 ], [ 5, 7 ], [ 5, 6 ],
  [ 5, 8 ], [ 5, 4 ], [ 6, 4 ], [ 6, 8 ], [ 6, 2 ], [ 6, 3 ],
  [ 7, 5 ], [ 7, 1 ], [ 7, 3 ], [ 7, 2 ], [ 8, 3 ], [ 8, 2 ],
  [ 8, 5 ], [ 8, 1 ] ]
gap> gr3 := DigraphRemoveEdgeOrbit(gr2, [1, 8]);
<immutable digraph with 8 vertices, 24 edges>
gap> gr3 = gr1;
true
```
3.3.24 DigraphRemoveEdges

▷ DigraphRemoveEdges(digraph, edges)  
   (operation)  
   Returns: A digraph.  
   If one of the following holds:  
   • digraph is a digraph with no multiple edges, and edges is a list of pairs of vertices of digraph,  
   or  
   • digraph is a digraph and edges is an empty list  
   then this operation returns a digraph constructed from digraph by removing all of the edges specified by edges (see DigraphRemoveEdge (3.3.22)).  
   If digraph belongs to IsMutableDigraph (3.1.2), then the edge is removed directly from digraph. If digraph belongs to IsImmutableDigraph (3.1.3), the edge is removed from an immutable copy of digraph and this new digraph is returned.  
   Note that if edges is empty, then this operation will always return digraph rather than a copy. Also, if any element of edges is invalid (i.e. does not define an edge of digraph) then that element will simply be ignored.

Example

```
gap> D := CycleDigraph(250000);  
<immutable cycle digraph with 250000 vertices>  
gap> D := DigraphRemoveEdges(D, [[250000, 1]]);  
<immutable digraph with 250000 vertices, 249999 edges>  
gap> D := DigraphMutableCopy(D);  
<mutable digraph with 250000 vertices, 249999 edges>  
gap> new := DigraphRemoveEdges(D, [[1, 2], [2, 3], [3, 100]]);  
<mutable digraph with 250000 vertices, 249997 edges>  
gap> new = D;  
true
```

3.3.25 DigraphRemoveLoops

▷ DigraphRemoveLoops(digraph)  
   (operation)  
   ▷ DigraphRemoveLoopsAttr(digraph)  
   (attribute)  
   Returns: A digraph.  
   If digraph is a digraph, then this operation returns a digraph constructed from digraph by removing every loop. A loop is an edge with equal source and range.  
   If digraph is immutable, then a new immutable digraph is returned. Otherwise, the loops are removed from the mutable digraph digraph in-place.

Example

```
gap> D := CycleDigraph(5);  
<immutable digraph with 5 vertices, 12 edges>  
gap> DigraphRemoveLoops(D);  
<immutable digraph with 5 vertices, 8 edges>  
gap> D := Digraph(IsMutableDigraph, [[1, 2], [1]]);  
<mutable digraph with 2 vertices, 3 edges>  
gap> DigraphRemoveLoops(D);  
<mutable digraph with 2 vertices, 2 edges>  
gap> D;  
<mutable digraph with 2 vertices, 2 edges>
```
3.3.26 DigraphRemoveAllMultipleEdges

\[ \text{DigraphRemoveAllMultipleEdges(digraph)} \]

\[ \text{DigraphRemoveAllMultipleEdgesAttr(digraph)} \]

**Returns:** A digraph.

If \textit{digraph} is a digraph, then this operation returns a digraph constructed from \textit{digraph} by removing all multiple edges. The result is the largest subdigraph of \textit{digraph} which does not contain multiple edges.

If \textit{digraph} is immutable, then a new immutable digraph is returned. Otherwise, the multiple edges of the mutable digraph \textit{digraph} are removed in-place.

Example

\[ \text{digraph} \]

\begin{verbatim}
gap> D1 := Digraph([[1, 2, 3, 2], [1, 1, 3], [2, 2, 2]]); <immutable multidigraph with 3 vertices, 10 edges> gap> D2 := DigraphRemoveAllMultipleEdges(D1); <immutable digraph with 3 vertices, 6 edges> gap> OutNeighbours(D2); [ [ 1, 2, 3 ], [ 1, 3 ], [ 2 ] ] gap> D := Digraph(IsMutableDigraph, [[2, 2], [1]]); <mutable multidigraph with 2 vertices, 3 edges> gap> DigraphRemoveAllMultipleEdges(D); <mutable digraph with 2 vertices, 2 edges> gap> D; <mutable digraph with 2 vertices, 2 edges>
\end{verbatim}

3.3.27 DigraphReverseEdges (for a digraph and a list of edges)

\[ \text{DigraphReverseEdges(digraph, edges)} \]

\[ \text{DigraphReverseEdge(digraph, edge)} \]

\[ \text{DigraphReverseEdge(digraph, src, ran)} \]

**Returns:** A digraph.

If \textit{digraph} is a digraph without multiple edges, and \textit{edges} is a list of pairs of vertices of \textit{digraph} (the entries of each pair corresponding to the source and the range of an edge, respectively), then \textit{DigraphReverseEdges} returns a digraph constructed from \textit{digraph} by reversing the orientation of every edge specified by \textit{edges}. If only one edge is to be reversed, then \textit{DigraphReverseEdge} can be used instead. In this case, the second argument should just be a single vertex-pair, or the second and third arguments should be the source and range of an edge respectively.

Note that even though \textit{digraph} cannot have multiple edges, the output may have multiple edges.

If \textit{digraph} belongs to \textit{IsMutableDigraph} (3.1.2), then the edges are reversed in \textit{digraph}. If \textit{digraph} belongs to \textit{IsImmutableDigraph} (3.1.3), an immutable copy of \textit{digraph} with the specified edges reversed is returned.

Example

\[ \text{digraph} \]

\begin{verbatim}
gap> D := DigraphFromDiSparse6String(".Tg?i@s?t_e?_qEsC"); <immutable digraph with 21 vertices, 8 edges> gap> DigraphEdges(D); [[ 1, 2 ], [ 1, 7 ], [ 1, 8 ], [ 5, 21 ], [ 7, 19 ], [ 9, 1 ], [ 11, 2 ], [ 21, 1 ]]
gap> new := DigraphReverseEdge(D, [7, 19]); <immutable digraph with 21 vertices, 8 edges> gap> DigraphEdges(new);
\end{verbatim}
Digraphs

\[
\begin{pmatrix}
[1, 2], [1, 7], [1, 8], [5, 21], [9, 1], [11, 2], \\
[19, 7], [21, 1]
\end{pmatrix}
\]

```gap
gap> D2 := DigraphMutableCopy(new);
 gap> new := DigraphReverseEdges(D2, [[19, 7]]);
_gap> D2 = new;
true
_gap> D = new;
true
```

### 3.3.28 DigraphDisjointUnion (for an arbitrary number of digraphs)

- **DigraphDisjointUnion**
  - (function)
  - **DigraphDisjointUnion(D1, D2, ...)**
  - **DigraphDisjointUnion(list)**

**Returns:** A digraph.

In the first form, if \(D1, D2, \text{etc.}\) are digraphs, then *DigraphDisjointUnion* returns their disjoint union. In the second form, if *list* is a non-empty list of digraphs, then *DigraphDisjointUnion* returns the disjoint union of the digraphs contained in the list.

For a disjoint union of digraphs, the vertex set is the disjoint union of the vertex sets, and the edge list is the disjoint union of the edge lists.

More specifically, for a collection of digraphs \(D1, D2, \ldots,\) the disjoint union with have \(\text{DigraphNrVertices}(D1) + \text{DigraphNrVertices}(D2) + \ldots\) vertices. The edges of \(D1\) will remain unchanged, whilst the edges of the \(i\)th digraph, \(D[i]\), will be changed so that they belong to the vertices of the disjoint union corresponding to \(D[i]\). In particular, the edges of \(D[i]\) will have their source and range increased by \(\text{DigraphNrVertices}(D1) + \ldots + \text{DigraphNrVertices}(D[i-1])\).

Note that previously set *DigraphVertexLabels* (5.1.10) will be lost.

If the first digraph \(D1[\text{list[1]}]\) belongs to *IsMutableDigraph* (3.1.2), then \(D1[\text{list[1]}]\) is modified in place to contain the appropriate vertices and edges. If 
*digraph* belongs to *IsImmutableDigraph* (3.1.3), a new immutable digraph containing the appropriate vertices and edges is returned.

**Example**

```gap
gap> D1 := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
gap> OutNeighbours(D1);
[ [ 2 ], [ 3 ], [ 1 ] ]
gap> D2 := CompleteDigraph(3);
<immutable complete digraph with 3 vertices>
gap> OutNeighbours(D2);
[ [ 2, 3 ], [ 1, 3 ], [ 1, 2 ] ]
gap> union := DigraphDisjointUnion(D1, D2);
<immutable digraph with 6 vertices, 9 edges>
gap> OutNeighbours(union);
[ [ 2 ], [ 3 ], [ 1 ], [ 5, 6 ], [ 4, 6 ], [ 4, 5 ] ]
```

### 3.3.29 DigraphEdgeUnion (for a positive number of digraphs)

- **DigraphEdgeUnion**
  - (function)
  - **DigraphEdgeUnion(D1, D2, ...)**
  - **DigraphEdgeUnion(list)**

**Returns:** A digraph.
In the first form, if $D_1$, $D_2$, etc. are digraphs, then `DigraphEdgeUnion` returns their edge union.
In the second form, if `list` is a non-empty list of digraphs, then `DigraphEdgeUnion` returns the edge union of the digraphs contained in the list.

The vertex set of the edge union of a collection of digraphs is the union of the vertex sets, whilst the edge list of the edge union is the concatenation of the edge lists. The number of vertices of the edge union is equal to the maximum number of vertices of one of the digraphs, whilst the number of edges of the edge union will equal the sum of the number of edges of each digraph.

Note that previously set `DigraphVertexLabels` (5.1.10) will be lost.

If the first digraph $D_1 [\text{list}[1]]$ belongs to `IsMutableDigraph` (3.1.2), then $D_1 [\text{list}[1]]$ is modified in place to contain the appropriate vertices and edges. If digraph belongs to `IsImmutableDigraph` (3.1.3), a new immutable digraph containing the appropriate vertices and edges is returned.

```gap
gap> D := CycleDigraph(10);
<immutable cycle digraph with 10 vertices>
gap> DigraphEdgeUnion(D, D);
<immutable multidigraph with 10 vertices, 20 edges>
gap> D1 := Digraph([[2], [1]]);
<immutable digraph with 2 vertices, 2 edges>
gap> D2 := Digraph([[2, 3], [2], [1]]);
<immutable digraph with 3 vertices, 4 edges>
gap> union := DigraphEdgeUnion(D1, D2);
<immutable multidigraph with 3 vertices, 6 edges>
gap> OutNeighbours(union);
[ [ 2, 2, 3 ], [ 1, 2 ], [ 1 ] ]
gap> union = DigraphByEdges(
> Concatenation(DigraphEdges(D1), DigraphEdges(D2)));
true
```

### 3.3.30 DigraphJoin (for a positive number of digraphs)

- `DigraphJoin(D1, D2, ...)`
- `DigraphJoin(list)`

**Returns:** A digraph.

In the first form, if $D_1$, $D_2$, etc. are digraphs, then `DigraphJoin` returns their join. In the second form, if `list` is a non-empty list of digraphs, then `DigraphJoin` returns the join of the digraphs contained in the list.

The join of a collection of digraphs $D_1$, $D_2$, ... is formed by first taking the `DigraphDisjointUnion` (3.3.28) of the collection. In the disjoint union, if $i \neq j$ then there are no edges between vertices corresponding to digraphs $D[i]$ and $D[j]$ in the collection; the join is created by including all such edges.

For example, the join of two empty digraphs is a complete bipartite digraph.

Note that previously set `DigraphVertexLabels` (5.1.10) will be lost.

If the first digraph $D_1 [\text{list}[1]]$ belongs to `IsMutableDigraph` (3.1.2), then $D_1 [\text{list}[1]]$ is modified in place to contain the appropriate vertices and edges. If digraph belongs to `IsImmutableDigraph` (3.1.3), a new immutable digraph containing the appropriate vertices and edges is returned.
Digraphs

Example

```gap
D := CompleteDigraph(3);
<immutable complete digraph with 3 vertices>
IsCompleteDigraph(DigraphJoin(D, D));
true
D2 := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
DigraphJoin(D, D2);
<immutable digraph with 6 vertices, 27 edges>
```

3.3.31 DigraphCartesianProduct (for a positive number of digraphs)

▷ DigraphCartesianProduct(gr1, gr2, ...)

Returns: A digraph.

In the first form, if gr1, gr2, etc. are digraphs, then DigraphCartesianProduct returns a digraph isomorphic to their Cartesian product.

In the second form, if list is a non-empty list of digraphs, then DigraphCartesianProduct returns a digraph isomorphic to the Cartesian product of the digraphs contained in the list.

Mathematically, the Cartesian product of two digraphs G, H is a digraph with vertex set Cartesian(DigraphVertices(G), DigraphVertices(H)) such that there is an edge from [u, u'] to [v, v'] iff u = v and there is an edge from u' to v' in H or u' = v' and there is an edge from u to v in G.

Due to technical reasons, the digraph D returned by DigraphCartesianProduct has vertex set [1 .. DigraphNrVertices(G)*DigraphNrVertices(H)] instead, and the exact method of encoding pairs of vertices into integers is implementation specific. The original vertex pair can be somewhat regained by using DigraphCartesianProductProjections (3.3.38). In addition, DigraphVertexLabels (5.1.10) are preserved: if vertex pair [u, u'] was encoded as i then the vertex label of i will be the pair of vertex labels of u and u' i.e. DigraphVertexLabel(D, i) = [DigraphVertexLabel(G, u), DigraphVertexLabel(H, u')].

As the Cartesian product is associative, the Cartesian product of a collection of digraphs gr1, gr2, ... is computed in the obvious fashion.

Example

```gap
gr := ChainDigraph(4);
<immutable chain digraph with 4 vertices>
gr2 := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
gr3 := DigraphCartesianProduct(gr, gr2);
<immutable digraph with 12 vertices, 21 edges>
IsIsomorphicDigraph(gr3,
[ [2, 5], [3, 6], [4, 7], [8],
  [6, 9], [7, 10], [8, 11], [12],
  [10, 1], [11, 2], [12, 3], [4]]));
true
```

3.3.32 DigraphDirectProduct (for a positive number of digraphs)

▷ DigraphDirectProduct(gr1, gr2, ...)

Returns: A digraph.

In the first form, if gr1, gr2, etc. are digraphs, then DigraphDirectProduct returns a digraph isomorphic to their direct product.

In the second form, if list is a non-empty list of digraphs, then DigraphDirectProduct returns a digraph isomorphic to the direct product of the digraphs contained in the list.

Mathematically, the direct product of two digraphs G, H is a digraph with vertex set Cartesian(DigraphVertices(G), DigraphVertices(H)) such that there is an edge from [u, u'] to [v, v'] iff there is an edge from u to v in H and there is an edge from u' to v in G.

Due to technical reasons, the digraph D returned by DigraphDirectProduct has vertex set [1 .. DigraphNrVertices(G)*DigraphNrVertices(H)] instead, and the exact method of encoding pairs of vertices into integers is implementation specific. The original vertex pair can be somewhat regained by using DigraphDirectProductProjections (3.3.38). In addition, DigraphVertexLabels (5.1.10) are preserved: if vertex pair [u, u'] was encoded as i then the vertex label of i will be the pair of vertex labels of u and u' i.e. DigraphVertexLabel(D, i) = [DigraphVertexLabel(G, u), DigraphVertexLabel(H, u')].

As the direct product is associative, the direct product of a collection of digraphs gr1, gr2, ... is computed in the obvious fashion.


**Returns:** A digraph.

In the first form, if $gr_1, gr_2, \ldots$ are digraphs, then DigraphDirectProduct returns a digraph isomorphic to their direct product.

In the second form, if $list$ is a non-empty list of digraphs, then DigraphDirectProduct returns a digraph isomorphic to the direct product of the digraphs contained in the list.

Mathematically, the direct product of two digraphs $G, H$ is a digraph with vertex set $\text{Cartesian}(\text{DigraphVertices}(G), \text{DigraphVertices}(H))$ such that there is an edge from $[u, u']$ to $[v, v']$ iff there is an edge from $u$ to $v$ in $G$ and an edge from $u'$ to $v'$ in $H$.

Due to technical reasons, the digraph $D$ returned by DigraphDirectProduct has vertex set $[1 \ldots \text{DigraphNrVertices}(G) \times \text{DigraphNrVertices}(H)]$ instead, and the exact method of encoding pairs of vertices into integers is implementation specific. The original vertex pair can be somewhat regained by using DigraphDirectProductProjections (3.3.39). In addition DigraphVertexLabels (5.1.10) are preserved: if vertex pair $[u, u']$ was encoded as $i$ then the vertex label of $i$ will be the pair of vertex labels of $u$ and $u'$ i.e. DigraphVertexLabel($D, i$) = Smallest Integer representation of DigraphVertexLabel($G, u$), DigraphVertexLabel($H, u'$)].

As the direct product is associative, the direct product of a collection of digraphs $gr_1, gr_2, \ldots$ is computed in the obvious fashion.

**Example**

```
gap> gr := ChainDigraph(4);
<immutable chain digraph with 4 vertices>
gap> gr2 := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
gap> gr3 := DigraphDirectProduct(gr, gr2);
<immutable digraph with 12 vertices, 9 edges>
gap> IsIsomorphicDigraph(gr3,
  Digraph([[6], [7], [8], []],
  [[10], [11], [12], []],
  [[2], [3], [4], []]));
true
```

### 3.3.33 ConormalProduct

**Returns:** A digraph.

If $D1$ and $D2$ are digraphs without multiple edges, then ConormalProduct calculates the co-normal product digraph (CNPD) of $D1$ and $D2$. CNPD has vertex set $V1 \times V2$ where $V1$ is the vertex set of $D1$ and $V2$ is the vertex set of $D2$ (a vertex $[a, b]$ has label $(a - 1) \times |V2| + b$ in the output). There is an edge from $[a, b]$ to $[c, d]$ when at least one of the following two conditions are satisfied:

- There is an edge from $a$ to $c$ in $D1$.
- There is an edge from $b$ to $d$ in $D2$.

**Example**

```
gap> ConormalProduct(DigraphSymmetricClosure(CycleDigraph(7)),
  DigraphSymmetricClosure(CycleDigraph(4)));
<immutable digraph with 28 vertices, 504 edges>
gap> ConormalProduct(NullDigraph(0), CompleteDigraph(10));
<immutable empty digraph with 0 vertices>
```
3.3.34 HomomorphicProduct

\textbf{HomomorphicProduct}(D1, D2) \hspace{0.5cm} \text{(operation)}

\textbf{Returns: A digraph.}

If D1 and D2 are digraphs without multiple edges, then HomomorphicProduct calculates the \textit{homomorphic product digraph} (HPD) of D1 and D2. HPD has vertex set V1 x V2 where V1 is the vertex set of D1 and V2 is the vertex set of D2 (a vertex \([a, b]\) has label \((a - 1) \times |V2| + b\) in the output). There is an edge from \([a, b]\) to \([c, d]\) when at least one of the following two conditions are satisfied:

- The vertices a and c of D1 are equal.
- There is an edge from a to c in D1 and no edge from b to d in D2.

Example

\begin{verbatim}
gap> HomomorphicProduct(PetersenGraph(), DigraphSymmetricClosure(ChainDigraph(4))); immutable digraph with 40 vertices, 460 edges
\end{verbatim}

\begin{verbatim}
gap> D1 := Digraph([ [2], [1, 3, 4], [2, 5], [2, 5], [3, 4] ]); immutable digraph with 5 vertices, 10 edges
\end{verbatim}

\begin{verbatim}
gap> D2 := Digraph([ [2], [1, 3], [2, 4], [3] ]); immutable digraph with 4 vertices, 6 edges
\end{verbatim}

\begin{verbatim}
gap> HomomorphicProduct(D1, D2); immutable digraph with 20 vertices, 180 edges
\end{verbatim}

3.3.35 LexicographicProduct

\textbf{LexicographicProduct}(D1, D2) \hspace{0.5cm} \text{(operation)}

\textbf{Returns: A digraph.}

If D1 and D2 are digraphs without multiple edges, then LexicographicProduct calculates the \textit{lexicographic product digraph} (LPD) of D1 and D2. LPD has vertex set V1 x V2 where V1 is the vertex set of D1 and V2 is the vertex set of D2 (a vertex \([a, b]\) has label \((a - 1) \times |V2| + b\) in the output). There is an edge from \([a, b]\) to \([c, d]\) when at least one of the following two conditions are satisfied:

- There is an edge from a to c in D1.
- The vertices a and c of D1 are equal and there is an edge from b to d in D2.

Example

\begin{verbatim}
gap> LexicographicProduct(DigraphSymmetricClosure(CycleDigraph(3)), DigraphSymmetricClosure(ChainDigraph(2))); immutable digraph with 6 vertices, 30 edges
\end{verbatim}

\begin{verbatim}
gap> OutNeighbours(last); [ [ 2, 3, 4, 5, 6 ], [ 1, 3, 4, 5, 6 ], [ 1, 2, 4, 5, 6 ], [ 1, 2, 3, 5, 6 ], [ 1, 2, 3, 4, 6 ], [ 1, 2, 3, 4, 5 ] ]
\end{verbatim}

\begin{verbatim}
gap> LexicographicProduct(NullDigraph(0), CompleteDigraph(10)); immutable empty digraph with 0 vertices
\end{verbatim}
3.3.36 ModularProduct

▷ ModularProduct(D1, D2)

Returns: A digraph.

If D1 and D2 are digraphs without multiple edges, then ModularProduct calculates the modular product digraph (MPD) of D1 and D2. MPD has vertex set V1 x V2 where V1 is the vertex set of D1 and V2 is the vertex set of D2 (a vertex [a, b] has label \((a - 1) \cdot |V2| + b\) in the output). There is an edge from \([a, b]\) to \([c, d]\) precisely when the following two conditions are satisfied:

• The vertices \(a\) and \(c\) of D1 are equal if and only if the vertices \(b\) and \(d\) of D2 are equal.

• There is an edge from \(a\) to \(c\) in D1 if and only if there is an edge from \(b\) to \(d\) in D2.

Notably, the complete (with loops) subdigraphs of MPD are precisely the partial isomorphisms from D1 to D2.

Example

```gap
> ModularProduct(Digraph([[1], [1, 2]]), Digraph([[1], [2]]));
<immutable digraph with 4 vertices, 4 edges>
> OutNeighbours(last);
[ [ 4 ], [ 2, 3 ], [ ], [ 4 ] ]
> ModularProduct(PetersenGraph(),
> DigraphSymmetricClosure(CycleDigraph(5)));
<immutable digraph with 50 vertices, 950 edges>
> ModularProduct(NullDigraph(0), CompleteDigraph(10));
<immutable empty digraph with 0 vertices>
```

3.3.37 StrongProduct

▷ StrongProduct(D1, D2)

Returns: A digraph.

If D1 and D2 are digraphs without multiple edges, then StrongProduct calculates the strong product digraph (SPD) of D1 and D2. SPD has vertex set V1 x V2 where V1 is the vertex set of D1 and V2 is the vertex set of D2 (a vertex \([a, b]\) has label \((a - 1) \cdot |V2| + b\) in the output). There is an edge from \([a, b]\) to \([c, d]\) when at least one of the following three conditions are satisfied:

• The vertices \(a\) and \(c\) of D1 are equal and there is an edge from \(b\) to \(d\) in D2.

• The vertices \(b\) and \(d\) of D2 are equal and there is an edge from \(a\) to \(c\) in D1.

• There is an edge from \(a\) to \(c\) in D1 and there is an edge from \(b\) to \(d\) in D2.

The SPD of two paths of lengths \(m\) and \(n\) is also the king’s graph for an \(m\) by \(n\) board.

Example

```gap
> D1 := Digraph([[2], [1, 3, 4], [2, 5], [2, 5], [3, 4]]);
<immutable digraph with 5 vertices, 10 edges>
> D2 := Digraph([[2], [1, 3, 4], [2], [2]]);
<immutable digraph with 4 vertices, 6 edges>
> StrongProduct(D1, D2);
<immutable digraph with 20 vertices, 130 edges>
> StrongProduct(DigraphSymmetricClosure(ChainDigraph(3)),
> DigraphSymmetricClosure(ChainDigraph(4)));
<immutable digraph with 12 vertices, 58 edges>
```
3.3.38 DigraphCartesianProductProjections

▷ DigraphCartesianProductProjections(digraph) (attribute)

Returns: A list of transformations.

If digraph is a Cartesian product digraph, digraph = DigraphCartesianProduct(gr_1, gr_2, ... ), then DigraphCartesianProductProjections returns a list proj such that proj[i] is the projection onto the i-th coordinate of the product.

A projection is an idempotent endomorphism of digraph. If gr1, gr2, ... are all loopless digraphs, then the induced subdigraph of digraph on the image of proj[i] is isomorphic to gr_i.

Currently this attribute is only set upon creating an immutable digraph via DigraphCartesianProduct and there is no way of calculating it for other digraphs.

For more information see DigraphCartesianProduct (3.3.31)

Example

```
gap> D := DigraphCartesianProduct(ChainDigraph(3), CycleDigraph(4),
  > Digraph([[2], [2]]));;
gap> HasDigraphCartesianProductProjections(D); true
gap> proj := DigraphCartesianProductProjections(D);; Length(proj);
  3
gap> IsIdempotent(proj[2]);
  true
gap> RankOfTransformation(proj[3]);
  2
gap> S := ImageSetOfTransformation(proj[2]);;
gap> IsIsomorphicDigraph(CycleDigraph(4), InducedSubdigraph(D, S));
  true
```

3.3.39 DigraphDirectProductProjections

▷ DigraphDirectProductProjections(digraph) (attribute)

Returns: A list of transformations.

If digraph is a direct product digraph, digraph = DigraphDirectProduct(gr_1, gr_2, ... ), then DigraphDirectProductProjections returns a list proj such that proj[i] is the projection onto the i-th coordinate of the product.

A projection is an idempotent endomorphism of digraph. If gr1, gr2, ... are all loopless digraphs, then the image of digraph under proj[i] is isomorphic to gr_i.

Currently this attribute is only set upon creating an immutable digraph via DigraphDirectProduct and there is no way of calculating it for other digraphs.

For more information, see DigraphDirectProduct (3.3.32)

Example

```
gap> D := DigraphDirectProduct(ChainDigraph(3), CycleDigraph(4),
  > Digraph([[2], [2]]));;
gap> HasDigraphDirectProductProjections(D); true
gap> proj := DigraphDirectProductProjections(D);; Length(proj);
  3
gap> IsIdempotent(proj[2]);
  true
gap> RankOfTransformation(proj[3]);
  2
```

Digraphs
Digraphs

```gap
gap> P := DigraphRemoveAllMultipleEdges(
> ReducedDigraph(OnDigraphs(D, proj[2])));
gap> IsIsomorphicDigraph(CycleDigraph(4), P); true
```

### 3.3.40 LineDigraph

- **Operation**
  - `LineDigraph(digraph)`
  - `EdgeDigraph(digraph)`

**Returns:** A digraph.

Given a digraph `digraph`, the operation returns the digraph obtained by associating a vertex with each edge of `digraph`, and creating an edge from a vertex `v` to a vertex `u` if and only if the terminal vertex of the edge associated with `v` is the start vertex of the edge associated with `u`.

Note that the returned digraph is always a new immutable digraph, and the argument `digraph` is never modified.

**Example**

```gap
gap> LineDigraph(CompleteDigraph(3));
<immutable digraph with 6 vertices, 12 edges>
gap> LineDigraph(ChainDigraph(3));
<immutable digraph with 2 vertices, 1 edge>
```

### 3.3.41 LineUndirectedDigraph

- **Operation**
  - `LineUndirectedDigraph(digraph)`
  - `EdgeUndirectedDigraph(digraph)`

**Returns:** A digraph.

Given a symmetric digraph `digraph`, the operation returns the symmetric digraph obtained by associating a vertex with each edge of `digraph`, ignoring directions and multiplicities, and adding an edge between two vertices if and only if the corresponding edges have a vertex in common.

Note that the returned digraph is always a new immutable digraph, and the argument `digraph` is never modified.

**Example**

```gap
gap> LineUndirectedDigraph(CompleteDigraph(3));
<immutable digraph with 3 vertices, 6 edges>
gap> LineUndirectedDigraph(DigraphSymmetricClosure(ChainDigraph(3)));
<immutable digraph with 2 vertices, 2 edges>
```

### 3.3.42 DoubleDigraph

- **Operation**
  - `DoubleDigraph(digraph)`

**Returns:** A digraph.

Let `digraph` be a digraph with vertex set `V`. This function returns the double digraph of `digraph`. The vertex set of the double digraph is the original vertex set together with a duplicate. The edges are `[u_1, v_2]` and `[u_2, v_1]` if and only if `[u, v]` is an edge in `digraph`, together with the original edges and their duplicates.

If `digraph` is mutable, then `digraph` is modified in-place. If `digraph` is immutable, then a new immutable digraph constructed as described above is returned.
3.3.43 BipartiteDoubleDigraph

**BipartiteDoubleDigraph(digraph)**

Returns: A digraph.

Let *digraph* be a digraph with vertex set *V*. This function returns the bipartite double digraph of *digraph*. The vertex set of the double digraph is the original vertex set together with a duplicate. The edges are *[u_1, v_2]* and *[u_2, v_1]* if and only if *[u, v]* is an edge in *digraph*. The resulting graph is bipartite, since the original edges are not included in the resulting digraph.

If *digraph* is mutable, then *digraph* is modified in-place. If *digraph* is immutable, then a new immutable digraph constructed as described above is returned.

```
gap> gamma := Digraph([[2], [3], [1]]);  # <immutable digraph with 3 vertices, 3 edges>
gap> DoubleDigraph(gamma);  # <immutable digraph with 6 vertices, 12 edges>
```

3.3.44 DigraphAddAllLoops

**DigraphAddAllLoops(digraph)**

**DigraphAddAllLoopsAttr(digraph)**

Returns: A digraph.

For a digraph *digraph* this operation returns a new digraph constructed from *digraph*, such that a loop is added for every vertex which did not have a loop in *digraph*.

If *digraph* is immutable, then a new immutable digraph is returned. Otherwise, the loops are added to the loopless vertices of the mutable digraph *digraph* in-place.

```
gap> D := EmptyDigraph(13);  # <immutable empty digraph with 13 vertices>
gap> D := DigraphAddAllLoops(D);  # <immutable reflexive digraph with 13 vertices, 13 edges>
gap> OutNeighbours(D);  # [ [ 1 ], [ 2 ], [ 3 ], [ 4 ], [ 5 ], [ 6 ], [ 7 ], [ 8 ], [ 9 ], [ 10 ], [ 11 ], [ 12 ], [ 13 ] ]
gap> D := Digraph([[1, 2, 3], [1, 3], [1]]);  # <immutable digraph with 3 vertices, 6 edges>
gap> D := DigraphAddAllLoops(D);  # <immutable reflexive digraph with 3 vertices, 8 edges>
```

```
gap> D := CycleDigraph(IsMutableDigraph, 3);  # <mutable digraph with 3 vertices, 3 edges>
gap> DigraphAddAllLoops(D);  # <mutable digraph with 3 vertices, 6 edges>
```
3.3.45 DistanceDigraph (for digraph and int)

\[ \text{DistanceDigraph(digraph, } i) \]  
\[ \text{DistanceDigraph(digraph, list)} \]

Returns: A digraph.

The first argument is a digraph, the second argument is a non-negative integer or a list of positive integers. This operation returns a digraph on the same set of vertices as `digraph`, with two vertices being adjacent if and only if the distance between them in `digraph` equals \( i \) or is a number in `list`. See DigraphShortestDistance (5.4.2).

If `digraph` is mutable, then `digraph` is modified in-place. If `digraph` is immutable, then a new immutable digraph constructed as described above is returned.

Example

```gap
gap> digraph := DigraphFromSparse6String(":Jn?AL'BC_DEbEF'GiaGHdIJeGKcKL_@McDHfILaBJfHMjKM");
<immutable symmetric digraph with 30 vertices, 90 edges>
gap> DistanceDigraph(digraph, 1);
<immutable digraph with 30 vertices, 90 edges>
gap> DistanceDigraph(digraph, [1, 2]);
<immutable digraph with 30 vertices, 270 edges>
```

3.3.46 DigraphClosure

\[ \text{DigraphClosure(digraph, } k) \]

Returns: A digraph.

Given a symmetric loopless digraph with no multiple edges `digraph`, the \( k \)-closure of `digraph` is defined to be the unique smallest symmetric loopless digraph `C` with no multiple edges on the vertices of `digraph` that contains all the edges of `digraph` and satisfies the property that the sum of the degrees of every two non-adjacent vertices in `C` is less than \( k \). See IsSymmetricDigraph (6.2.14), DigraphHasLoops (6.2.1), IsMultiDigraph (6.2.11), and OutDegreeOfVertex (5.2.10).

The operation DigraphClosure returns the \( k \)-closure of `digraph`.

Example

```gap
gap> D := CompleteDigraph(6);
<immutable complete digraph with 6 vertices>
gap> D := DigraphRemoveEdges(D, [[1, 2], [2, 1]]);
<immutable digraph with 6 vertices, 28 edges>
gap> closure := DigraphClosure(D, 6);
<immutable digraph with 6 vertices, 30 edges>
gap> IsCompleteDigraph(closure);
true
```

3.3.47 DigraphMycielskian

\[ \text{DigraphMycielskian(digraph)} \]
\[ \text{DigraphMycielskianAttr(digraph)} \]

Returns: A digraph.
If *digraph* is a symmetric digraph, then DigraphMycielskian returns the Mycielskian of *digraph*.

The Mycielskian of a symmetric digraph is a larger symmetric digraph constructed from it, which has a larger chromatic number. For further information, see https://en.wikipedia.org/wiki/Mycielskian.

If *digraph* is immutable, then a new immutable digraph is returned. Otherwise, the mutable digraph *digraph* is changed in-place into its Mycielskian.

```
gap> D := CycleDigraph(2);  
<immutable cycle digraph with 2 vertices>  
gap> ChromaticNumber(D);  
2  
gap> D := DigraphMycielskian(D);  
<immutable digraph with 5 vertices, 10 edges>  
gap> ChromaticNumber(D);  
3  
gap> D := DigraphMycielskian(D);  
<immutable digraph with 11 vertices, 40 edges>  
gap> ChromaticNumber(D);  
4  
gap> D := CompleteBipartiteDigraph(IsMutable, 2, 3);  
<mutable digraph with 5 vertices, 12 edges>  
gap> DigraphMycielskian(D);  
<mutable digraph with 11 vertices, 46 edges>  
gap> D;  
<mutable digraph with 11 vertices, 46 edges>
```

### 3.4 Random digraphs

#### 3.4.1 RandomDigraph

D RandomDigraph([filt, ]n[, p])

**Returns:** A digraph.

If the optional first argument *filt* is present, then this should specify the category or representation the digraph being created will belong to. For example, if *filt* is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if *filt* is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument *filt* is not present, then IsImmutableDigraph (3.1.3) is used by default.

The other implemented filters are as follows: IsConnectedDigraph (6.6.3), IsSymmetricDigraph (6.2.14), IsAcyclicDigraph (6.6.1), IsEulerianDigraph (6.6.10), IsHamiltonianDigraph (6.6.11).

For IsConnectedDigraph (6.6.3), a random tree is first created independent of the value of *p*, guaranteeing connectivity (with *n* − 1 edges), and then edges are added between the remaining pairs of vertices with probability approximately *p*.

For IsHamiltonianDigraph (6.6.11), a random Hamiltonian cycle is first created independent of the value of *p* (with *n* edges), and then edges are added between the remaining pairs of vertices with probability approximately *p*.
For IsEulerianDigraph (6.6.10), a random Eulerian cycle is created where \( p \) influences how long the cycle will be. The cycle grows by randomly considering edges that extend the cycle, and adding an edge with probability approximately \( p \). The cycle stops when we get back to the start vertex and have no more edges left to consider from it that extend the cycle further (any possible edge from the start vertex has either been added to the cycle, or rejected, leaving no more edges to consider). Thus \( p = 1 \) does not necessarily guarantee a complete digraph. Instead, it guarantees that all edges considered up to the point where the cycle stops, are added.

For IsAcyclicDigraph (6.6.1) and IsSymmetricDigraph (6.2.14), edges are added between any pairs of vertices with probability approximately \( p \).

If \( n \) is a positive integer, then this function returns a random digraph with \( n \) vertices and without multiple edges. The result may or may not have loops. If using IsAcyclicDigraph (6.6.1), the resulting graph will not have any loops by definition.

If the optional second argument \( p \) is a float with value \( 0 \leq p \leq 1 \), then an edge will exist between each pair of vertices with probability approximately \( p \). If \( p \) is not specified, then a random probability will be assumed (chosen with uniform probability).

Example:
```
gap> RandomDigraph(1000);  
<immutable digraph with 1000 vertices, 364444 edges>  
gap> RandomDigraph(1000, 0.023);  
<immutable digraph with 10000 vertices, 2300438 edges>  
gap> RandomDigraph(IsMutableDigraph, 1000, 1 / 2);  
<mutable digraph with 1000 vertices, 499739 edges>  
gap> RandomDigraph(IsConnectedDigraph, 1000, 0.75);  
<immutable digraph with 1000 vertices, 750265 edges>  
gap> RandomDigraph(IsSymmetricDigraph, 1000);  
<immutable digraph with 1000 vertices, 329690 edges>  
gap> RandomDigraph(IsAcyclicDigraph, 1000, 0.25);  
<immutable digraph with 1000 vertices, 125070 edges>  
gap> RandomDigraph(IsHamiltonianDigraph, 1000, 0.5);  
<immutable digraph with 1000 vertices, 500327 edges>  
gap> RandomDigraph(IsEulerianDigraph, 1000, 0.5);  
<immutable digraph with 1000 vertices, 433869 edges>
```

### 3.4.2 RandomMultiDigraph

```
▷ RandomMultiDigraph(\([n, m]\))  (operation)  

Returns: A digraph.

If \( n \) is a positive integer, then this function returns a random digraph with \( n \) vertices. If the optional second argument \( m \) is a positive integer, then the digraph will have \( m \) edges. If \( m \) is not specified, then the number of edges will be chosen randomly (with uniform probability) from the range \([1 .. \binom{n}{2}]\).

The method used by this function chooses each edge from the set of all possible edges with uniform probability. No effort is made to avoid creating multiple edges, so it is possible (but not guaranteed) that the result will have multiple edges. The result may or may not have loops.

Example:
```
gap> RandomMultiDigraph(1000);  
<immutable multidigraph with 1000 vertices, 216659 edges>  
gap> RandomMultiDigraph(1000, 950);  
<immutable multidigraph with 1000 vertices, 950 edges>
```
3.4.3 RandomTournament

▷ RandomTournament([filt, ]n) (operation)

Returns: A digraph.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

If n is a non-negative integer, this function returns a random tournament with n vertices. See IsTournament (6.2.15).

Example
\[
\text{gap> RandomTournament(10);}
<\text{immutable tournament with 10 vertices}>
\]
\[
\text{gap> RandomTournament(IsMutableDigraph, 10);}
<\text{mutable digraph with 1000 vertices, 500601 edges}>
\]

3.4.4 RandomLattice

▷ RandomLattice(n) (operation)

Returns: A digraph.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

If n is a positive integer, this function return a random lattice with m vertices, where it is guaranteed that m is between n and 2 * n. See IsLatticeDigraph (6.4.3).

Example
\[
\text{gap> RandomLattice(10);}
<\text{immutable lattice digraph with 10 vertices, 39 edges}>
\]
\[
\text{gap> RandomLattice(IsMutableDigraph, 10);}
<\text{mutable digraph with 12 vertices, 52 edges}>
\]

3.5 Standard examples

3.5.1 AndrasfaiGraph

▷ AndrasfaiGraph([filt, ]n) (operation)

Returns: A digraph.

If n is an integer greater than 0, then this operation returns the nth Andrasfai graph. The Andrasfai graph is a circulant graph with 3n - 1 vertices. The indices of the Andrasfai graph are given by the numbers between 1 and 3n - 1 that are congruent to 1 mod 3 (that is, for each index j, vertex i is adjacent to the i + jth and i - j vertices). The graph has 6(3n - 1) edges. The graph is triangle free.

As a circulant graph, the Andrasfai graph is biconnected, cyclic, Hamiltonian, regular, and vertex transitive.

See https://mathworld.wolfram.com/AndrasfaiGraph.html for further details.
If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> D := AndrasfaiGraph(4);  
<immutable Hamiltonian biconnected vertex-transitive symmetric digraph\ 
with 11 vertices, 44 edges>
gap> IsBiconnectedDigraph(D);  
true
gap> IsIsomorphicDigraph(D, CirculantGraph(11, [1, 4, 7, 10]));
true
```

### 3.5.2 BananaTree

Successors to the operation

**BananaTree([filt, ]n, k)**

**Returns:** A digraph.

If n and k are positive integers with k greater than 1, then this operation returns the banana tree with parameters n and k, as defined below.

- From [https://mathworld.wolfram.com/BananaTree.html](https://mathworld.wolfram.com/BananaTree.html):
  - “An (n,k)-banana tree, as defined by Chen et al. (1997), is a graph obtained by connecting one leaf of each of n copies of an k-star graph with a single root vertex that is distinct from all the stars.”

Specifically, in the resulting digraph, vertex 1 is the 'root', and for each m in [1 .. k], the mth star is on the vertices [(m - 1) * n + 2 .. (m * n) + 1], with the first of these being the 'centre' of the star, and the second being the 'leaf' adjacent 1.

See also StarGraph (3.5.43) and IsUndirectedTree (6.6.9).

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> D := BananaTree(2, 4);  
<immutable undirected tree digraph with 9 vertices>
gap> D := BananaTree(3, 3);  
<immutable undirected tree digraph with 10 vertices>
gap> D := BananaTree(5, 2);  
<immutable undirected tree digraph with 11 vertices>
gap> D := BananaTree(3, 4);  
<immutable undirected tree digraph with 13 vertices>
```

### 3.5.3 BinaryTree

Successors to the operation

**BinaryTree([filt, ]m)**

**Returns:** A digraph.

This function returns a binary tree of depth m with \(2^m - 1\) vertices. All edges are directed towards the root of the tree, which is vertex 1.
Note that BinaryTree(m) is the induced subdigraph of BinaryTree(m+1) on the vertices \([1..2^{-(m-1)}]\).

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

\[
\begin{align*}
\text{Example} \\
gap> \text{BinaryTree(1)}; \\
<\text{immutable empty digraph with 1 vertex}> \\
gap> \text{BinaryTree(8)}; \\
<\text{immutable digraph with 255 vertices, 254 edges}> \\
gap> \text{BinaryTree(IsMutableDigraph, 8)}; \\
<\text{mutable digraph with 255 vertices, 254 edges}>
\end{align*}
\]

### 3.5.4 BinomialTreeGraph

\[\text{BinomialTreeGraph([filt, ]n)}\]

**Returns:** A digraph.

If \(n\) is a positive integer then this operation returns the \(n\)th binomial tree graph. The binomial tree graph has \(n\) vertices and \(n-1\) undirected edges. The vertices of the binomial tree graph are the numbers from 1 to \(n\) in binary representation, with a vertex \(v\) having as a direct parent the vertex with binary representation the same as \(v\) but with the lowest 1-bit cleared. For example, the vertex 011 has parent 010, and the vertex 010 has parent 000.

The binomial tree graph is an undirected tree, and is symmetric as a digraph.

See [https://metacpan.org/pod/Graph::Maker::BinomialTree](https://metacpan.org/pod/Graph::Maker::BinomialTree) for further details.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

\[
\begin{align*}
\text{Example} \\
gap> D := \text{BinomialTreeGraph(9)}; \\
<\text{immutable undirected tree digraph with 9 vertices}>
\end{align*}
\]

### 3.5.5 BishopsGraph

\[\text{BishopsGraph([filt, ]color, ]m, n)}\]

\[\text{BishopGraph([filt, ]color, ]m, n)}\]

**Returns:** A digraph.

If \(m\) and \(n\) are positive integers, then this operation returns the bishop’s graph of an \(m\) by \(n\) chessboard, as a symmetric digraph.

A bishop’s graph represents all possible moves of the bishop chess piece across a chessboard. An \(m\) by \(n\) chessboard is a grid of \(m\) columns (‘files’) and \(n\) rows (‘ranks’) that intersect in squares. Orthogonally adjacent squares are alternately colored light and dark, with the square in the first rank and file being dark.
The \( m \times n \) vertices of the bishop’s graph can be placed onto the \( m \times n \) squares of an \( m \) by \( n \) chessboard, such that two vertices are adjacent in the digraph if and only if a bishop can move between the corresponding squares in a single turn. A legal bishop’s move is defined as any move which moves the bishop piece to a diagonally adjacent square or to a square which can be reached through a series of diagonally adjacent squares, with all of these small moves being in the same direction.

The chosen correspondence between vertices and chess squares is given by \texttt{DigraphVertexLabels} (5.1.10). In more detail, the vertices of the digraph are labelled by elements of the Cartesian product \([1..m] \times [1..n]\), where the first entry indexes the column (file) of the square in the chessboard, and the second entry indexes the row (rank) of the square. (Note that the files are traditionally indexed by the lowercase letters of the alphabet). The vertices are sorted in ascending order, first by row (second component) and then column (first component).

If the optional second argument \texttt{color} is present, then this should be one of the strings "dark", "light", or "both". The default is "both". A bishop on a light square can only move to light squares, and a bishop on a dark square can only move to dark squares. This optional argument controls which bishops are represented in the resulting digraph. If \texttt{color} is "both", then the resulting digraph will show all the vertices of an \( m \) by \( n \) chessboard, and will be disconnected (unless \( m = n = 1 \)). Otherwise, \texttt{BishopsGraph} returns the induced subdigraph of this on the vertices that correspond to either the dark squares or the light squares, according to the value of \texttt{color}.

If the optional first argument \texttt{filt} is present, then this should specify the category or representation the digraph being created will belong to. For example, if \texttt{filt} is \texttt{IsMutableDigraph} (3.1.2), then the digraph being created will be mutable, if \texttt{filt} is \texttt{IsImmutableDigraph} (3.1.3), then the digraph will be immutable. If the optional first argument \texttt{filt} is not present, then \texttt{IsImmutableDigraph} (3.1.3) is used by default.

```gap
gap> BishopsGraph(8, 8);
<immutable symmetric digraph with 64 vertices, 560 edges>
gap> D := BishopsGraph("dark", 3, 5);
<immutable connected symmetric digraph with 8 vertices, 24 edges>
gap> IsConnectedDigraph(D);
true
gap> BishopsGraph("light", 4, 4);
<immutable connected symmetric digraph with 8 vertices, 28 edges>
gap> D := BishopsGraph("both", 1, 5);
<immutable empty digraph with 5 vertices>
```

### 3.5.6 BondyGraph

\texttt{BondyGraph([filt, ]n)}

**Returns:** A digraph.

If \( n \) is a non-negative integer then this operation returns the \( n \)th Bondy graph. The Bondy graphs are a family of hypohamiltonian graphs: a graph which is not Hamiltonian itself but the removal of any vertex produces a Hamiltonian graph. The Bondy graphs are the \((3(2n + 1) + 2, 2)\)-th Generalised Petersen graphs, and have \(12n + 10\) vertices and \(15 + 18n\) undirected edges.

See https://mathworld.wolfram.com/HypohamiltonianGraph.html for further details.

If the optional first argument \texttt{filt} is present, then this should specify the category or representation the digraph being created will belong to. For example, if \texttt{filt} is \texttt{IsMutableDigraph} (3.1.2), then the digraph being created will be mutable, if \texttt{filt} is \texttt{IsImmutableDigraph} (3.1.3),
then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> D := BondyGraph(1);
<immutable symmetric digraph with 22 vertices, 66 edges>
gap> IsHamiltonianDigraph(D);
false
gap> G := List([1 .. 22], x -> DigraphRemoveVertex(D, x));;

gap> ForAll(G, IsHamiltonianDigraph);
true
```

### 3.5.7 BookGraph

▷ BookGraph([filt, ]m)

(operation)

Returns: A digraph.

The book graph is the Cartesian product of a complete digraph with two vertices (as the "book spine") and the m + 1 star graph (as the "pages"). For more details on the book graph please refer to https://mathworld.wolfram.com/BookGraph.html.

See DigraphCartesianProduct (3.3.31), CompleteDigraph (3.5.11), and StarGraph (3.5.43).

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> BookGraph(1);
<immutable bipartite symmetric digraph with bicomponent sizes 2 and 2>
gap> BookGraph(2);
<immutable bipartite symmetric digraph with bicomponent sizes 3 and 3>
gap> BookGraph(IsMutable, 12);
<mutable digraph with 26 vertices, 74 edges>
gap> BookGraph(7);
<immutable bipartite symmetric digraph with bicomponent sizes 8 and 8>
gap> IsSymmetricDigraph(BookGraph(24));
true
gap> IsBipartiteDigraph(BookGraph(24));
true
```

### 3.5.8 StackedBookGraph

▷ StackedBookGraph([filt, ]m, n)

(operation)

Returns: A digraph.

The stacked book graph is the Cartesian product of the symmetric closure of the chain digraph with n vertices (as the "book spine") and the m + 1 star graph (as the "pages"). For more details on the stacked book graph please refer to https://mathworld.wolfram.com/StackedBookGraph.html.

See DigraphCartesianProduct (3.3.31), DigraphSymmetricClosure (3.3.12), ChainDigraph (3.5.9), and StarGraph (3.5.43).

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph
(3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
Example

gap> StackedBookGraph(1, 1);
<immutable bipartite symmetric digraph with bicomponent sizes 1 and 1>
gap> StackedBookGraph(1, 2);
<immutable bipartite symmetric digraph with bicomponent sizes 2 and 2>
gap> StackedBookGraph(3, 4);
<immutable bipartite symmetric digraph with bicomponent sizes 8 and 8>
gap> StackedBookGraph(IsMutable, 12, 5);
<mutable digraph with 65 vertices, 224 edges>
gap> StackedBookGraph(5, 5);
<immutable bipartite symmetric digraph with bicomponent sizes 13 and 17>
gap> IsSymmetricDigraph(StackedBookGraph(24, 8));
true
gap> IsBipartiteDigraph(StackedBookGraph(24, 8));
true
```

### 3.5.9 ChainDigraph

**ChainDigraph([filt, ]n)**

**Returns:** A digraph.

If \( n \) is a positive integer, this function returns a chain with \( n \) vertices and \( n - 1 \) edges. Specifically, for each vertex \( i \) (with \( i < n \)), there is a directed edge with source \( i \) and range \( i + 1 \).

The DigraphReflexiveTransitiveClosure (3.3.13) of a chain represents a total order.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
Example

gap> ChainDigraph(42);
<immutable chain digraph with 42 vertices>
gap> ChainDigraph(IsMutableDigraph, 10);
<mutable digraph with 10 vertices, 9 edges>
```

### 3.5.10 CirculantGraph

**CirculantGraph([filt, ]n, par)**

**Returns:** A digraph.

If \( n \) is an integer greater than 1, and par is a list of integers that are contained in \([1..n]\) then this operation returns a *circulant graph*. The circulant graph is a graph on \( n \) vertices, where for each element \( j \) of par, the \( i \)th vertex is adjacent to the \((i + j)\)th and \((i - j)\)th vertices.

If par is \([1]\), then the graph is the \( n \)th cyclic graph. If par is \([1,2,\ldots,\text{Int}(n/2)]\), then the graph is the complete graph on \( n \) vertices. If \( n \) is at least 4 and par is \([1,n]\) then the graph is the \( n \)th Mobius ladder graph.
A circulant graph is biconnected, cyclic, Hamiltonian, regular, and vertex transitive. See https://mathworld.wolfram.com/CirculantGraph.html for further details.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```
gap> D := CirculantGraph(6, [1, 2, 3]);
<immutable Hamiltonian biconnected vertex-transitive symmetric digraph with 6 vertices, 30 edges>
gap> IsIsomorphicDigraph(D, CompleteDigraph(6));
true
```

### 3.5.11 CompleteDigraph

**Definition:**

```
CompleteDigraph([filt, ]n)
```

**Returns:**

A digraph.

If n is a non-negative integer, this function returns the complete digraph with n vertices. See IsCompleteDigraph (6.2.5).

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```
gap> CompleteDigraph(20);
<immutable complete digraph with 20 vertices>
gap> CompleteDigraph(IsMutableDigraph, 10);
<mutable digraph with 10 vertices, 90 edges>
```

### 3.5.12 CompleteBipartiteDigraph

**Definition:**

```
CompleteBipartiteDigraph([filt, ]m, n)
```

**Returns:**

A digraph.

A complete bipartite digraph is a digraph whose vertices can be partitioned into two non-empty vertex sets, such there exists a unique edge with source i and range j if and only if i and j lie in different vertex sets.

If m and n are positive integers, this function returns the complete bipartite digraph with vertex sets of sizes m (containing the vertices [1 .. m]) and n (containing the vertices [m + 1 .. m + n]).

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```
gap> CompleteBipartiteDigraph(20, 10);
```
3.5.13 CompleteMultipartiteDigraph

 Returns: A digraph.

 For a list \( \text{orders} \) of \( n \) positive integers, this function returns the digraph containing \( n \) independent sets of vertices of orders \( [l_1 \ldots l_n] \). Moreover, each vertex is adjacent to every other not contained in the same independent set.

 If the optional first argument \( \text{filt} \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( \text{filt} \) is \text{IsMutableDigraph} (3.1.2), then the digraph being created will be mutable, if \( \text{filt} \) is \text{IsImmutableDigraph} (3.1.3), then the digraph will be immutable. If the optional first argument \( \text{filt} \) is not present, then \text{IsImmutableDigraph} (3.1.3) is used by default.

 Example

\[
\begin{array}{l}
gap> \text{CompleteMultipartiteDigraph([5, 4, 2]);} \\
<\text{immutable complete multipartite digraph with 11 vertices, 76 edges}> \\
gap> \text{CompleteMultipartiteDigraph(IsMutableDigraph, [5, 4, 2]);} \\
<\text{mutable digraph with 11 vertices, 76 edges}> \\
\end{array}
\]

3.5.14 CycleDigraph

 Returns: A digraph.

 If \( n \) is a positive integer, then these functions return a cycle digraph with \( n \) vertices and \( n \) edges. Specifically, for each vertex \( i \) (with \( i < n \)), there is a directed edge with source \( i \) and range \( i + 1 \). In addition, there is an edge with source \( n \) and range \( 1 \).

 If the optional first argument \( \text{filt} \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( \text{filt} \) is \text{IsMutableDigraph} (3.1.2), then the digraph being created will be mutable, if \( \text{filt} \) is \text{IsImmutableDigraph} (3.1.3), then the digraph will be immutable. If the optional first argument \( \text{filt} \) is not present, then \text{IsImmutableDigraph} (3.1.3) is used by default.

 Example

\[
\begin{array}{l}
gap> \text{CycleDigraph(1);} \\
<\text{immutable digraph with 1 vertex, 1 edge}> \\
gap> \text{CycleDigraph(123);} \\
<\text{immutable cycle digraph with 123 vertices}> \\
gap> \text{CycleDigraph(IsMutableDigraph, 10);} \\
<\text{mutable digraph with 10 vertices, 10 edges}> \\
gap> \text{DigraphCycle(4) = CycleDigraph(4);} \\
true \\
\end{array}
\]
3.5.15 CycleGraph

▷ CycleGraph([filt, ]n)  
    Returns: A digraph.

If \( n \) is an integer greater than 2 then this operation returns the \( n \)th cycle graph, consisting of the cycle on \( n \) vertices. The cycle graph, unlike the cycle digraph, is symmetric. The cycle graph has \( n \) vertices and \( n \) undirected edges. The cycle graph is simple so the non-simple graphs with a single vertex and single loop and with two vertices and two edges between them are excluded.

See https://mathworld.wolfram.com/CycleGraph.html for further details.

If the optional first argument \( filt \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( filt \) is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if \( filt \) is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument \( filt \) is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> D := CycleGraph(7);
<immutable symmetric digraph with 7 vertices, 14 edges>
```

3.5.16 EmptyDigraph

▷ EmptyDigraph([filt, ]n)  
▷ NullDigraph([filt, ]n)  
    Returns: A digraph.

If \( n \) is a non-negative integer, this function returns the empty or null digraph with \( n \) vertices. An empty digraph is one with no edges.

NullDigraph is a synonym for EmptyDigraph.

If the optional first argument \( filt \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( filt \) is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if \( filt \) is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument \( filt \) is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> EmptyDigraph(20);
<immutable empty digraph with 20 vertices>
gap> NullDigraph(10);
<immutable empty digraph with 10 vertices>
gap> EmptyDigraph(IsMutableDigraph, 10);
<mutable empty digraph with 10 vertices>
```

3.5.17 GearGraph

▷ GearGraph([filt, ]n)  
    Returns: A digraph.

If \( n \) is a positive integer at least 3, then this operation returns the gear graph with \( 2n + 1 \) vertices and \( 3n \) undirected edges. The \( n \)th gear graph is the \( 2 \)th cycle graph with one additional central vertex, to which every other vertex of the cycle is connected. The gear graph is a symmetric digraph. A gear graph is a Matchstick graph, that is it is simple with a planar graph embedding, and is a unit-distance
graph (that is, it can be embedded into the Euclidean plane with vertices being distinct points and edges having length 1).

See https://mathworld.wolfram.com/GearGraph.html for further details.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> D := GearGraph(4);
<immutable symmetric digraph with 9 vertices, 24 edges>
gap> ChromaticNumber(D);
2
gap> IsVertexTransitive(D);
false
```

#### 3.5.18 HaarGraph

▸ HaarGraph([filt, ]n)

**Returns:** A digraph.

If n is a positive integer then this operation returns the Haar graph $H(n)$.

The number of vertices in the Haar graph $H(n)$ is equal to twice $m$, where $m$ is the number of digits required to represent $n$ in binary. These vertices are arranged into bicomponents $[1..m]$ and $[m+1..2*m]$. Vertices $i$ and $j$ in different bicomponents are adjacent by a symmetric pair of edges if and only if the binary representation of $n$ has a 1 in position $(j-i$ modulo $m$) + 1 from the left.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> HaarGraph(3);
<immutable bipartite vertex-transitive symmetric digraph with bicomponent sizes 2 and 2>
```

#### 3.5.19 HalvedCubeGraph

▸ HalvedCubeGraph([filt, ]n)

**Returns:** A digraph.

If n is a positive integer at least 1, then this operation returns the nth halved cube graph, the graph of the $n$-demihypercube. The vertices of the graph are those of the $n$-hypercube, with two vertices adjacent if and only if they are at distance 1 or 2 from each other. Equivalent constructions are as the second graph power of the $n-1$ hypercube graph, or as with vertices labelled as the binary numbers where two vertices are adjacent if they differ in a single bit, or with vertices labelled with the subset of
binary numbers with even Hamming weight, with edges connecting vertices with Hamming distance exactly 2. The Halved Cube graph is distance regular and contains a Hamiltonian cycle.

See https://mathworld.wolfram.com/HalvedCubeGraph.html for further details.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```
gap> D := HalvedCubeGraph(3);
<immutable Hamiltonian connected symmetric digraph with 4 vertices, 12 edges>
gap> IsDistanceRegularDigraph(D);
true
gap> IsHamiltonianDigraph(D);
true
```

3.5.20 HanoiGraph

▷ HanoiGraph([filt, ]n) (operation)

Returns: A digraph.

If n is positive integer then this operation returns the nth Hanoi graph. The Hanoi graph’s vertices represent the possible states of the 'Tower of Hanoi' puzzle on three 'towers', while its edges represent possible moves. The Hanoi graph has $3^n$ vertices, and $\binom{3}{3} \times (3^n - 1) / 2$ undirected edges.

The Hanoi graph is Hamiltonian. The graph superficially resembles the Sierpinski triangle. The graph is also a 'penny graph' - a graph whose vertices can be considered to be non-overlapping unit circles on a flat surface, with two vertices adjacent only if the unit circles touch at a single point. Thus the Hanoi graph is planar.

See https://mathworld.wolfram.com/HanoiGraph.html for further details.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```
gap> D := HanoiGraph(5);
<immutable Hamiltonian connected symmetric digraph with 243 vertices, 726 edges>
gap> IsPlanarDigraph(D);
true
```

3.5.21 HelmGraph

▷ HelmGraph([filt, ]n) (operation)

Returns: A digraph.
If \( n \) is a positive integer at least 3, then this operation returns the \( n \)th helm graph. The helm graph is the \( n-1 \)th wheel graph with, for each external vertex of the 'wheel', adjoining a new vertex incident only to the first vertex. That is, the graph looks similar to a ship's helm.

See https://mathworld.wolfram.com/WheelGraph.html for further details.

If the optional first argument \( \text{filt} \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( \text{filt} \) is \( \text{IsMutableDigraph} \) (3.1.2), then the digraph being created will be mutable, if \( \text{filt} \) is \( \text{IsImmutableDigraph} \) (3.1.3), then the digraph will be immutable. If the optional first argument \( \text{filt} \) is not present, then \( \text{IsImmutableDigraph} \) (3.1.3) is used by default.

Example

\[ \text{gap> D := HelmGraph(4);} \]
\[ \text{<immutable symmetric digraph with 9 vertices, 24 edges>} \]

### 3.5.22 HypercubeGraph

▷ HypercubeGraph([\( \text{filt} \), \( n \)]) (operation)

Returns: A digraph.

If \( n \) is a non-negative integer, then this operation returns the \( n \)th hypercube graph. The graph has \( 2^n \) vertices and \( 2^{n-1} \) edges. It is formed from the vertices and edges of the \( n \)-dimensional hypercube. Alternatively, the graph can be constructed by labelling each vertex with the binary numbers, with two vertices adjacent if they have Hamming distance exactly one. The hypercube graphs are Hamiltonian, distance-transitive and therefore distance-regular, and bipartite.

See https://mathworld.wolfram.com/HypercubeGraph.html for further details.

If the optional first argument \( \text{filt} \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( \text{filt} \) is \( \text{IsMutableDigraph} \) (3.1.2), then the digraph being created will be mutable, if \( \text{filt} \) is \( \text{IsImmutableDigraph} \) (3.1.3), then the digraph will be immutable. If the optional first argument \( \text{filt} \) is not present, then \( \text{IsImmutableDigraph} \) (3.1.3) is used by default.

Example

\[ \text{gap> D := HypercubeGraph(5);} \]
\[ \text{<immutable Hamiltonian connected bipartite symmetric digraph with bicomponent sizes 16 and 16>} \]

### 3.5.23 JohnsonDigraph

▷ JohnsonDigraph([\( \text{filt} \), \( n \), \( k \)]) (operation)

Returns: A digraph.

If \( n \) and \( k \) are non-negative integers, then this operation returns a symmetric digraph which corresponds to the undirected Johnson graph \( J(n,k) \).

The Johnson graph \( J(n,k) \) has vertices given by all the \( k \)-subsets of the range \([1 \ldots n]\), and two vertices are connected by an edge if and only if their intersection has size \( k-1 \).

If the optional first argument \( \text{filt} \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( \text{filt} \) is \( \text{IsMutableDigraph} \) (3.1.2), then the digraph being created will be mutable, if \( \text{filt} \) is \( \text{IsImmutableDigraph} \) (3.1.3), then the digraph will be immutable. If the optional first argument \( \text{filt} \) is not present, then \( \text{IsImmutableDigraph} \) (3.1.3) is used by default.
Digraphs

Example

```gap
gap> gr := JohnsonDigraph(3, 1);
<immutable symmetric digraph with 3 vertices, 6 edges>
gap> OutNeighbours(gr);
[ [ 2, 3 ], [ 1, 3 ], [ 1, 2 ] ]
gap> gr := JohnsonDigraph(4, 2);
<immutable symmetric digraph with 6 vertices, 24 edges>
gap> OutNeighbours(gr);
[ [ 2, 3, 4, 5 ], [ 1, 3, 4, 6 ], [ 1, 2, 5, 6 ],
  [ 1, 3, 4, 6 ], [ 2, 3, 4, 5 ] ]
gap> JohnsonDigraph(1, 0);
<immutable empty digraph with 1 vertex>
gap> JohnsonDigraph(IsMutableDigraph, 1, 0);
<mutable empty digraph with 1 vertex>
```

3.5.24 KellerGraph

▷ KellerGraph([[filt, ]n])

Returns: A digraph.

If n is a nonnegative integer then this operation returns the nth or n-dimensional Keller graph. The graph has vertices given by the n-tuples on the set \([0,1,2,3]\). Two vertices are adjacent if their respective tuples are such that they differ in at least two coordinates and in at least one coordinate the difference between the two is \(2 \mod 4\). The Keller graph has \(4^n\) vertices.

The Keller graphs were constructed with the intention of finding counterexamples to Keller’s conjecture (https://mathworld.wolfram.com/KellersConjecture.html), and has been used since for testing maximum clique algorithms.

If n is 1 then the graph is empty, for n greater than 1 the chromatic number of the Keller graph is \(2^n\) and the graph is Hamiltonian.

See https://mathworld.wolfram.com/KellerGraph.html for further details.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

```gap
gap> D := KellerGraph(3);
<immutable Hamiltonian connected symmetric digraph with 64 vertices, 2\176 edges>
gap> ChromaticNumber(D);
8
```

3.5.25 KingsGraph

▷ KingsGraph([[filt, ]m, n])

Returns: A digraph.

If m and n are positive integers, then this operation returns the king’s graph of an m by n chessboard, as a symmetric digraph.
The king’s graph represents all possible moves of the king chess piece across a chessboard. An \( m \times n \) chessboard is a grid of \( m \) columns (‘files’) and \( n \) rows (‘ranks’) that intersect in squares. Orthogonally adjacent squares are alternately colored light and dark, with the square in the first rank and file being dark.

The king can move only to any orthogonally or diagonally adjacent square. Thus the \( m \times n \) vertices of the king’s graph can be placed onto the \( m \times n \) squares of an \( m \times n \) chessboard, such that two vertices are adjacent in the digraph if and only if the corresponding squares are orthogonally or diagonally adjacent on the chessboard.

The chosen correspondence between vertices and chess squares is given by `DigraphVertexLabels (5.1.10)`. In more detail, the vertices of the digraph are labelled by elements of the Cartesian product \([1..m] \times [1..n]\), where the first entry indexes the column (file) of the square in the chessboard, and the second entry indexes the row (rank) of the square. (Note that the files are traditionally indexed by the lowercase letters of the alphabet). The vertices are sorted in ascending order, first by row (second component) and then column (first component).

See Wikipedia for further information. See also `SquareGridGraph (3.5.41)`, `TriangularGridGraph (3.5.42)`, and `StrongProduct (3.3.37)`.

If the optional first argument \( \text{filt} \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( \text{filt} \) is `IsMutableDigraph (3.1.2)`, then the digraph being created will be mutable, if \( \text{filt} \) is `IsImmutableDigraph (3.1.3)`, then the digraph will be immutable. If the optional first argument \( \text{filt} \) is not present, then `IsImmutableDigraph (3.1.3)` is used by default.

```gap
gap> KingsGraph(8, 8);
<immutable connected symmetric digraph with 64 vertices, 420 edges>
gap> D := KingsGraph(IsMutable, 2, 7);
<mutable digraph with 14 vertices, 62 edges>
gap> IsPlanarDigraph(D);
true
gap> D := KingsGraph(3, 3);
<immutable connected symmetric digraph with 9 vertices, 40 edges>
gap> OutNeighbors(D);
[ [ 2, 4, 5 ], [ 1, 3, 5, 4, 6 ], [ 2, 6, 5 ], [ 5, 1, 7, 2, 8 ],
 [ 4, 6, 2, 8, 3, 7, 1, 9 ], [ 5, 3, 9, 8, 2 ], [ 8, 4, 5 ],
 [ 7, 9, 5, 6, 4 ], [ 8, 6, 5 ] ]
gap> IsSubdigraph(QueensGraph(3, 4), KingsGraph(3, 4));
true
```

### 3.5.26 KneserGraph

\( \text{KneserGraph}(\text{filt}, \text{In}, \text{k}) \)

**Returns:** A digraph.

If \( \text{n} \) and \( \text{k} \) are integers greater than 0, with \( \text{k} \) less than \( \text{n} \), then this operation returns the \((\text{n}, \text{k})\)-th Kneser graph. The Kneser graph’s vertices correspond to the \( \text{k} \)-element subsets of a set of \( \text{n} \) elements, with two vertices being adjacent if and only if the subsets are disjoint. The graph has \( \text{Binomial}(\text{n}, \text{k}) \) vertices and \( \text{Binomial}(\text{n}, \text{k}) \times \text{Binomial}(\text{n} - \text{k}, \text{k}) \) edges. Kneser graphs are regular, edge transitive, and vertex transitive. If \( \text{k} \) is 1, then the graph is the complete graph on \( \text{n} \) vertices. If \((\text{n}, \text{k})\) is \((2\text{m}-1, \text{n}-1)\), then the graph is the \( \text{m} \)th Odd graph. The Petersen graph is the \((5,2)\)th Kneser graph.
If \( n \geq 2k \) then the graph’s chromatic number is \( n - 2k + 2 \), and otherwise is 1. The Kneser graph contains a Hamiltonian cycle if \( n \geq ((3 + 5^{-0.5})/2)k + 1 \). The graph has clique number equal to the floor of \( n / k \).

See https://mathworld.wolfram.com/KneserGraph.html for further details.

If the optional first argument \( \text{filt} \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( \text{filt} \) is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if \( \text{filt} \) is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument \( \text{filt} \) is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

\[
\text{gap> } D := \text{KneserGraph}(7, 3);
<\text{immutable edge- and vertex-transitive symmetric digraph with 35 vertices, 140 edges}>
\]
\[
\text{gap> } \text{IsRegularDigraph}(D);
\text{true}
\]
\[
\text{gap> } \text{ChromaticNumber}(D);
3
\]
\[
\text{gap> } \text{CliqueNumber}(D);
2
\]

### 3.5.27 KnightsGraph

\( \triangleright \text{KnightsGraph([filt, ]m, n)} \) (operation)

**Returns:** A digraph.

If \( m \) and \( n \) are positive integers, then this operation returns the knight’s graph of an \( m \) by \( n \) chessboard, as a symmetric digraph.

A knight’s graph represents all possible moves of the knight chess piece across a chessboard. An \( m \) by \( n \) chessboard is a grid of \( m \) columns (‘files’) and \( n \) rows (‘ranks’) that intersect in squares. Orthogonally adjacent squares are alternately colored light and dark, with the square in the first rank and file being dark.

The \( m \times n \) vertices of the knight’s graph can be placed onto the \( m \times n \) squares of an \( m \) by \( n \) chessboard, such that two vertices are adjacent in the digraph if and only if a knight can move between the corresponding squares in a single turn.

The chosen correspondence between vertices and chess squares is given by DigraphVertexLabels (5.1.10). In more detail, the vertices of the digraph are labelled by elements of the Cartesian product \([1..m] \times [1..n]\), where the first entry indexes the column (file) of the square in the chessboard, and the second entry indexes the row (rank) of the square. (Note that the files are traditionally indexed by the lowercase letters of the alphabet). The vertices are sorted in ascending order, first by row (second component) and then column (first component).

See Wikipedia for further information.

If the optional first argument \( \text{filt} \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( \text{filt} \) is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if \( \text{filt} \) is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument \( \text{filt} \) is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

\[
\text{gap> } D := \text{KnightsGraph}(8, 8);
<\text{immutable connected symmetric digraph with 64 vertices, 336 edges}>
\]
3.5.28 LindgrenSousselierGraph

\[ \text{LindgrenSousselierGraph}(\text{filt}, \text{n}) \]

**Returns:** A digraph.

If \( n \) is an integer greater than 0, then this operation returns the \( n \)th Lindgren-Sousselier graph, an infinite family of hypohamiltonian graphs - a graph that is non-Hamiltonian but removing any vector gives a Hamiltonian graph. The graph has \( 6n+4 \) vertices and \( 15 + 10(n-1) \) undirected edges. The first Lindgren-Sousselier graph is the Petersen graph, and is in fact the smallest hypohamiltonian graph.

See [https://mathworld.wolfram.com/HypohamiltonianGraph.html](https://mathworld.wolfram.com/HypohamiltonianGraph.html) for further details.

If the optional first argument \( \text{filt} \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( \text{filt} \) is \( \text{IsMutableDigraph} \) (3.1.2), then the digraph being created will be mutable, if \( \text{filt} \) is \( \text{IsImmutableDigraph} \) (3.1.3), then the digraph will be immutable. If the optional first argument \( \text{filt} \) is not present, then \( \text{IsImmutableDigraph} \) (3.1.3) is used by default.

\[
gap> D := \text{LindgrenSousselierGraph}(3); \\
<\text{immutable symmetric digraph with 22 vertices, 70 edges}> \\
gap> \text{IsHamiltonianDigraph}(D); \\
false \\
gap> \text{IsHamiltonianDigraph}(\text{DigraphRemoveVertex}(D, 1)); \\
true \\
gap> \text{IsIsomorphicDigraph}(\text{LindgrenSousselierGraph}(1), \text{PetersenGraph}()); \\
true
\]

---

3.5.29 LollipopGraph

\[ \text{LollipopGraph}([\text{filt}, \text{m, n}]) \]

**Returns:** A digraph.

If \( m \) and \( n \) are positive integers, then this operation returns the \((m, n)\)-lollipop graph. As defined at [https://en.wikipedia.org/wiki/Lollipop_graph](https://en.wikipedia.org/wiki/Lollipop_graph), this consists of a complete digraph on the vertices \([1..m]\) (the 'head' of the lollipop), and the symmetric closure of a chain digraph on the remaining \( n \) vertices (the 'stick'), connected by a bridge (the edge \([m, m+1]\) and its reverse).

If the optional first argument \( \text{filt} \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( \text{filt} \) is \( \text{IsMutableDigraph} \) (3.1.2), then the digraph being created will be mutable, if \( \text{filt} \) is \( \text{IsImmutableDigraph} \) (3.1.3), then the digraph will be immutable. If the optional first argument \( \text{filt} \) is not present, then \( \text{IsImmutableDigraph} \) (3.1.3) is used by default.

\[
gap> D := \text{LollipopGraph}(3, 9); \\
<\text{mutable digraph with 27 vertices, 88 edges}> \\
gap> \text{IsConnectedDigraph}(D); \\
true \\
gap> \text{IsConnectedDigraph}(\text{KnightsGraph}(3, 3)); \\
false \\
gap> \text{KnightsGraph}(\text{IsMutable}, 3, 9); \\
<\text{mutable digraph with 27 vertices, 88 edges}>
\]
Digraphs

Example

```gap
D := LollipopGraph(5, 3);
<immutable connected symmetric digraph with 8 vertices, 26 edges>
CliqueNumber(D);
5
DigraphUndirectedGirth(D);
3
LollipopGraph(IsMutableDigraph, 3, 8);
<mutable digraph with 11 vertices, 22 edges>
```

3.5.30 MobiusLadderGraph

▷ MobiusLadderGraph([filt, ]n)

Returns: A digraph.

If n is a positive integer at least 4, then this operation returns the *Mobius ladder graph* that is obtained by introducing a 'twist' in the n-th prism graph, similar to the construction of a Mobius strip. The Mobius ladder graph is isomorphic to the circulant graph $C_2(2n, [1, n])$. The Mobius ladders are cubic, symmetric, Hamiltonian, vertex-transitive, and graceful. They are also non-planar and apex, meaning removing a single vertex produces a planar graph.

See [https://mathworld.wolfram.com/MoebiusLadder.html](https://mathworld.wolfram.com/MoebiusLadder.html) for further details.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

```gap
D := MobiusLadderGraph(7);
<immutable symmetric digraph with 14 vertices, 42 edges>
IsHamiltonianDigraph(D);
true
IsVertexTransitive(D);
true
IsPlanarDigraph(D);
false
D2 := DigraphRemoveVertex(D, 1);
<immutable digraph with 13 vertices, 36 edges>
IsPlanarDigraph(D2);
true
```

3.5.31 MycielskiGraph

▷ MycielskiGraph([filt, ]n)

Returns: A digraph.

If n is an integer greater than 1, then this operation returns the n-th *Mycielski graph*. The Mycielskian of a triangle-free graph is a construction that adds vertices and edges to produce a new graph that is still triangle-free but has a larger chromatic number. The Mycielski graphs are a series of graphs with this construction repeated, starting with the complete graph on two vertices. The graph has $3 \cdot 2^{(n-2)} - 1$ vertices, with the number of edges being $(18 \cdot 27 \cdot 2^{-n} + 14 \cdot 3^{-n}) / 36$. 
The Mycielski graph has chromatic number equal to \( n \), clique number equal to 2, and is Hamiltonian. The graph is in fact the graph with chromatic number \( n \) with the least possible vertices.

See [https://mathworld.wolfram.com/MycielskiGraph.html](https://mathworld.wolfram.com/MycielskiGraph.html) for further details.

If the optional first argument \( filt \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( filt \) is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if \( filt \) is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument \( filt \) is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> D := MycielskiGraph(4);
<immutable Hamiltonian connected symmetric digraph with 11 vertices, 40 edges>
gap> ChromaticNumber(D);
4
gap> CliqueNumber(D);
2
```

### 3.5.32 OddGraph

\begin{verbatim}
▷ OddGraph([filt, ]n)

Returns: A digraph.
\end{verbatim}

If \( n \) is an integer greater than 0, then this operation returns the \( n \)th odd graph. The odd graph has vertices labelled with the \( n-1 \)-subsets of a \( 2n-1 \)-set, with two vertices adjacent if and only if their subsets are disjoint. The \( n \)th odd graph is the \((2n-1, n-1)\)-th Kneser graph. The graph has \( \text{Binomial}(2n-1, n-1) \) vertices and \( n \times \text{Binomial}(2n-1, n-1) / 2 \) edges.

The odd graph is regular and distance transitive (and therefore distance regular). They have chromatic number equal to 3, and all Odd graphs with \( n \) greater than 3 are Hamiltonian. They are also vertex and edge transitive.

See [https://mathworld.wolfram.com/OddGraph.html](https://mathworld.wolfram.com/OddGraph.html) for further details.

If the optional first argument \( filt \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( filt \) is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if \( filt \) is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument \(filt \) is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> D := OddGraph(4);
<immutable edge- and vertex-transitive symmetric digraph with 35 vertices, 140 edges>
gap> IsIsomorphicDigraph(D, KneserGraph(7, 3));
true
gap> ChromaticNumber(D);
3
```

### 3.5.33 PathGraph

\begin{verbatim}
▷ PathGraph([filt, ]n)

Returns: A digraph.
\end{verbatim}
If \( n \) is a non-negative integer then this operation returns the \( n \)th path graph, consisting of the path on \( n \) vertices. This is the symmetric closure of the ChainDigraph (3.5.9). The path graph has \( n \) vertices and \( n - 1 \) edges. The path graph is an undirected tree.

See [https://mathworld.wolfram.com/PathGraph.html](https://mathworld.wolfram.com/PathGraph.html) for further details.

If the optional first argument \( \text{filt} \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( \text{filt} \) is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if \( \text{filt} \) is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument \( \text{filt} \) is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
D := PathGraph(12);
<immutable undirected tree digraph with 12 vertices>
```

### 3.5.34 PermutationStarGraph

▷ **PermutationStarGraph([\( \text{filt} \), \( \text{n} \), \( \text{k} \)]**)  

**Returns:** A digraph.

If \( n \) is an integer at greater than 0 and \( k \) is an integer greater than 1, and \( k \) less than or equal to \( n \), then this operation returns the \((\text{n}, \; \text{k})\)-th permutation star graph. The vertices of the graph are given by the \( k \)-length ordered subsets of an \( n \)-set, with two vertices being adjacent if one is labelled \( p_1 \; p_2 \; p_3 \ldots \; p_i \ldots \; p_k \), and the other is either labelled \( p_i \; p_2 \; p_3 \ldots \; p_1 \ldots \; p_k \), or labelled \( x \; p_2 \; p_3 \ldots \; p_i \ldots \; p_k \) where \( x \) is in \([1..n]\) and is not equal to \( p_1 \). The graph has \( n! / (n - k)! \) vertices.

The permutation star graph is regular and vertex transitive. It has diameter \( 2k - 1 \) if \( k \) less than or equal to \( \text{Int}(n / 2) \), and diameter \( \text{Int}((n - 1) / 2) + k \) if \( k \geq \text{Int}(n / 2) + 1 \).

See [https://mathworld.wolfram.com/PermutationStarGraph.html](https://mathworld.wolfram.com/PermutationStarGraph.html) for further details.

If the optional first argument \( \text{filt} \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( \text{filt} \) is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if \( \text{filt} \) is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument \( \text{filt} \) is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
D := PermutationStarGraph(6, 2);
<immutable vertex-transitive symmetric digraph with 30 vertices, 150 edges>

Diameter(D);
3
```

### 3.5.35 PetersenGraph

▷ **PetersenGraph([\( \text{filt} \)])**  

**Returns:** A digraph.

From [https://en.wikipedia.org/wiki/Petersen_graph](https://en.wikipedia.org/wiki/Petersen_graph):

“The Petersen graph is an undirected graph with 10 vertices and 15 edges. It is a small graph that serves as a useful example and counterexample for many problems in graph theory. The Petersen graph is named after Julius Petersen, who in 1898 constructed it to be the smallest bridgeless cubic graph with no three-edge-coloring.”
See also GeneralisedPetersenGraph (3.5.36).

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```
gap> ChromaticNumber(PetersenGraph());
3
gap> PetersenGraph(IsMutableDigraph);
<mutable digraph with 10 vertices, 30 edges>
```

3.5.36 GeneralisedPetersenGraph

▷ GeneralisedPetersenGraph([filt, ]n, k)  

Returns: A digraph.

If n is a positive integer and k is a non-negative integer less than \( n / 2 \), then this operation returns the generalised Petersen graph \( GPG(n,k) \).

From https://en.wikipedia.org/wiki/Generalized_Petersen_graph:

“The generalized Petersen graphs are a family of cubic graphs formed by connecting the vertices of a regular polygon to the corresponding vertices of a star polygon.

They include the Petersen graph and generalize one of the ways of constructing the Petersen graph. The generalized Petersen graph family was introduced in 1950 by H. S. M. Coxeter and was given its name in 1969 by Mark Watkins.”

See also PetersenGraph (3.5.35).

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```
gap> GeneralisedPetersenGraph(7, 2);
<immutable symmetric digraph with 14 vertices, 42 edges>
gap> GeneralisedPetersenGraph(40, 1);
<immutable symmetric digraph with 80 vertices, 240 edges>
gap> D := GeneralisedPetersenGraph(5, 2);
<immutable symmetric digraph with 10 vertices, 30 edges>
gap> IsIsomorphicDigraph(D, PetersenGraph());
true
```

3.5.37 PrismGraph

▷ PrismGraph([filt, ]n)  

Returns: A digraph.

If n is a positive integer at least 3, then this operation returns the prism graph that is the skeleton of the \( n \)-prism. It has \( 2n \) vertices and \( 3n \) undirected edges. The prism graph is a symmetric digraph. The \( n \)th prism graph is isomorphic to the graph Cartesian product of the second path graph and the
nth cycle graph, isomorphic to the generalised Petersen graph GP(n,1). If n is odd then the prism graph is isomorphic to the Circulant graph C1(2n, [2,n]). The prism graph is Hamiltonian.

See https://mathworld.wolfram.com/PrismGraph.html for further details.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```
gap> D := PrismGraph(4);
<immutable symmetric digraph with 8 vertices, 24 edges>
gap> IsHamiltonianDigraph(D);
true
gap> D := PrismGraph(5);
<immutable symmetric digraph with 10 vertices, 30 edges>
gap> IsIsomorphicDigraph(D, CirculantGraph(10, [2, 5]));
true
```

3.5.38 StackedPrismGraph

▷ StackedPrismGraph([filt, ]n, k)

Returns: A digraph.

If n is an integer at least 3 and k is a positive integer then this operation returns the (n,k)-th stacked prism graph. The graph is k concentric n-Cycle graphs connected by spokes. The stacked prism is the graph Cartesian product of the n-th cycle graph and the k-th path graph. The graph has nk vertices and n(2k - 1) undirected edges. If k is 1 then the graph is the n-th cycle graph, if k is 2 then the graph is the prism graph.

See https://mathworld.wolfram.com/StackedPrismGraph.html for further details.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```
gap> D := StackedPrismGraph(5, 2);
<immutable symmetric digraph with 10 vertices, 30 edges>
gap> IsIsomorphicDigraph(D, PrismGraph(5));
true
```

3.5.39 QueensGraph

▷ QueensGraph([filt, ]m, n)

▷ QueenGraph([filt, ]m, n)

Returns: A digraph.

If m and n are positive integers, then this operation returns the queen’s graph of an m by n chessboard, as a symmetric digraph.

The queen’s graph represents all possible moves of the queen chess piece across a chessboard. An m by n chessboard is a grid of m columns (‘files’) and n rows (‘ranks’) that intersect in squares.
Orthogonally adjacent squares are alternately colored light and dark, with the square in the first rank and file being dark.

The \( m \times n \) vertices of the queen’s graph can be placed onto the \( m \times n \) squares of an \( m \times n \) chessboard, such that two vertices are adjacent in the digraph if and only if the queen can move between the corresponding squares in a single turn. A legal queen’s move is defined as one which moves the queen to an (orthogonally or diagonally) adjacent square, or to a square which can be reached through a series of such moves, with all of the small moves being in the same direction.

Note that the QueensGraph is the DigraphEdgeUnion (3.3.29) of the RooksGraph (3.5.40) and the BishopsGraph (3.5.5) of the same dimensions.

The chosen correspondence between vertices and chess squares is given by DigraphVertexLabels (5.1.10). In more detail, the vertices of the digraph are labelled by elements of the Cartesian product \([1..m] \times [1..n]\), where the first entry indexes the column (file) of the square in the chessboard, and the second entry indexes the row (rank) of the square. (Note that the files are traditionally indexed by the lowercase letters of the alphabet). The vertices are sorted in ascending order, first by row (second component) and then column (first component).

If the optional first argument \( \text{filt} \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( \text{filt} \) is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if \( \text{filt} \) is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument \( \text{filt} \) is not present, then IsImmutableDigraph (3.1.3) is used by default.

```gap
gap> QueensGraph(2, 5);
<immutable connected symmetric digraph with 10 vertices, 66 edges>
gap> D := QueensGraph(4, 3);
<immutable connected symmetric digraph with 12 vertices, 92 edges>
gap> IsRegularDigraph(D);
false
gap> QueensGraph(6, 9) = DigraphEdgeUnion(RooksGraph(6, 9), BishopsGraph(6, 9));
true
```

### Example

#### 3.5.40 RooksGraph

\[ \text{RooksGraph}([\text{filt}, \text{m}, \text{n}]) \]

\[ \text{RookGraph}([\text{filt}, \text{m}, \text{n}]) \]

**Returns:** A digraph.

If \( m \) and \( n \) are positive integers, then this operation returns the rook’s graph of an \( m \times n \) chessboard, as a symmetric digraph.

A rook’s graph represents all possible moves of the rook chess piece across a chessboard. An \( m \times n \) chessboard is a grid of \( m \) columns (‘files’) and \( n \) rows (‘ranks’) that intersect in squares. Orthogonally adjacent squares are alternately colored light and dark, with the square in the first rank and file being dark.

The \( m \times n \) vertices of the rook’s graph can be placed onto the \( m \times n \) squares of an \( m \times n \) chessboard, such that two vertices are adjacent in the digraph if and only if a rook can move between the corresponding squares in a single turn. A legal rook’s move is defined as one which moves the rook to an orthogonally adjacent square, or to a square which can be reached through a series of such moves, with all of the small moves being in the same direction.
The chosen correspondence between vertices and chess squares is given by DigraphVertexLabels (5.1.10). In more detail, the vertices of the digraph are labelled by elements of the Cartesian product \([1..m] \times [1..n]\), where the first entry indexes the column (file) of the square in the chessboard, and the second entry indexes the row (rank) of the square. (Note that the files are traditionally indexed by the lowercase letters of the alphabet). The vertices are sorted in ascending order, first by row (second component) and then column (first component).

See Wikipedia for further information. See also DigraphCartesianProduct (3.3.31) and LineDigraph (3.3.40).

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
gap> D := RooksGraph(7, 4);
<immutable connected regular symmetric digraph with 28 vertices, 252 edges>
gap> RooksGraph(1, 8);
<immutable connected regular symmetric digraph with 8 vertices, 56 edges>
```

### 3.5.41 SquareGridGraph

- **SquareGridGraph**([filt, ]n, k)  
- **GridGraph**([filt, ]n, k)

**Returns:** A digraph.

If \(n\) and \(k\) are positive integers, then this operation returns a square grid graph of dimension \(n\) by \(k\).

A square grid graph of dimension \(n\) by \(k\) is the DigraphCartesianProduct (3.3.31) of the symmetric closures of the chain digraphs with \(n\) and \(k\) vertices; see DigraphSymmetricClosure (3.3.12) and ChainDigraph (3.5.9).

In particular, the \(n \times k\) vertices can be arranged into an \(n\) by \(k\) grid such that two vertices are adjacent in the digraph if and only if they are orthogonally adjacent in the grid. The correspondence between vertices and grid positions is given by DigraphVertexLabels (5.1.10).

See [https://en.wikipedia.org/wiki/Lattice_graph](https://en.wikipedia.org/wiki/Lattice_graph) for more information.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

```
gap> SquareGridGraph(5, 5);
<immutable connected bipartite symmetric digraph with bicomponent sizes 13 and 12>
gap> GridGraph(IsMutable, 3, 4);
<mutable digraph with 12 vertices, 34 edges>
```
3.5.42 TriangularGridGraph

▷ TriangularGridGraph([filt, ]n, k) (operation)

Returns: A digraph.

If \( n \) and \( k \) are positive integers, then this operation returns a triangular grid graph of dimension \( n \) by \( k \).

A triangular grid graph of dimension \( n \) by \( k \) is a symmetric digraph constructed from the SquareGridGraph (3.5.41) of the same dimensions, where additionally two vertices are adjacent in the digraph if they are diagonally adjacent in the grid, on a particular one of the diagonals. The correspondence between vertices and grid positions is given by DigraphVertexLabels (5.1.10). More specifically, the particular diagonal is the one such that, the vertices corresponding to the grid positions \([2,1]\) and \([1,2]\) are adjacent (if they exist), but those corresponding to \([1,1]\) and \([2,2]\) are not.

See https://en.wikipedia.org/wiki/Lattice_graph#Other_kinds for more information.

If the optional first argument \( filt \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( filt \) is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if \( filt \) is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument \( filt \) is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

\[
\text{gap> TriangularGridGraph(3, 3);}\\
\text{<immutable connected symmetric digraph with 9 vertices, 32 edges>}\\
\text{gap> TriangularGridGraph(IsMutable, 3, 3);}\\
\text{<mutable digraph with 9 vertices, 32 edges>}
\]

3.5.43 StarGraph

▷ StarGraph([filt, ]k) (operation)

Returns: A digraph.

If \( k \) is a positive integer, then this operation returns the star graph with \( k \) vertices, which is the undirected tree in which vertex 1 is adjacent to all other vertices. If \( k \) is at least 2, then this is the complete bipartite digraph with bicomponents \([1]\) and \([2 \ldots k]\).

See IsUndirectedTree (6.6.9), IsCompleteBipartiteDigraph (6.2.4), and DigraphBicomponents (5.4.13).

If the optional first argument \( filt \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( filt \) is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if \( filt \) is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument \( filt \) is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

\[
\text{gap> StarGraph(IsMutable, 10);}\\
\text{<mutable digraph with 10 vertices, 18 edges>}\\
\text{gap> StarGraph(5);}\\
\text{<immutable complete bipartite digraph with bicomponent sizes 1 and 4>}\\
\text{gap> IsSymmetricDigraph(StarGraph(3));}\\
\text{true}\\
\text{gap> IsUndirectedTree(StarGraph(3));}\\
\text{true}
\]
3.5.44 TadpoleGraph

▷ TadpoleGraph([filt, ]m, n)

(operation)

Returns: A digraph.

The tadpole graph is the symmetric closure of the disjoint union of the cycle digraph on \([1..m]\) (the 'head' of the tadpole) and the chain digraph on \([m+1..m+n]\) (the 'tail' of the tadpole), along with the additional edges \([1, m+1]\) and \([1, m+1]\) which connect the 'head' and the 'tail'. For more details on the tadpole graph please refer to https://en.wikipedia.org/wiki/Tadpole_graph.

See DigraphSymmetricClosure (3.3.12), DigraphDisjointUnion (3.3.28), CycleDigraph (3.5.14), and ChainDigraph (3.5.9).

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> TadpoleGraph(10, 15);
<immutable symmetric digraph with 25 vertices, 50 edges>
gap> TadpoleGraph(IsMutableDigraph, 5, 6);
<mutable digraph with 11 vertices, 22 edges>
gap> IsSymmetricDigraph(TadpoleGraph(3, 5));
true
```

3.5.45 WalshHadamardGraph

▷ WalshHadamardGraph([filt, ]n)

(operation)

Returns: A digraph.

If \(n\) is a positive integer at least 1, then this operation returns the Hadamard graph constructed from the \(n\)th Hadamard matrix (of dimension \(2^n\)) as constructed by Joseph Walsh. A Hadamard matrix is a square matrix with entries either 1 or \(-1\), such that all the rows are mutually orthogonal. The \(n\)th Walsh Hadamard graph is a graph on \(4^n\) matrices split into four categories \(r_i+, r_i-, c_i+, c_i-\).

If \(h_{ij}\) are the elements of the \(n\)th Walsh matrix, then if \(h_{ij} = 1\) then \((r_i+, c_j+)\) and \((r_i-, c_j-)\) are edges, if \(h_{ij} = -1\) then \((r_i+, c_j-)\) and \((r_i-, c_j+)\) are edges. Walsh Hadamard graphs are distance transitive and distance regular.

See https://mathworld.wolfram.com/HadamardGraph.html for further details.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> D := WalshHadamardGraph(5);
<immutable symmetric digraph with 64 vertices, 1024 edges>
gap> IsDistanceRegularDigraph(D);
true
```
3.5.46 WebGraph

▷ WebGraph([filt, ]n)

Returns: A digraph.

If n is an integer at least 3 then this operation returns the n-th web graph. The web graph is the $(n, 3)$-th stacked prism graph with the edges of the outer cycle removed. The graph has $3n$ vertices and $4n$ undirected edges.

See https://mathworld.wolfram.com/WebGraph.html for further details.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> D := WebGraph(5);
<immutable symmetric digraph with 15 vertices, 40 edges>
```

3.5.47 WheelGraph

▷ WheelGraph([filt, ]n)

Returns: A digraph.

If n is a positive integer at least 4, then this operation returns the n-th wheel graph. The Wheel graph is formed from an $n-1$ cycle graph with a single additional vertex adjacent to all vertices of the cycle. The graph has $n$ vertices and $2(n-1)$ edges. Wheel graphs are the skeletons of $n-1$ pyramids, and are self-dual. If $n$ is odd, then the chromatic number of the wheel graph is 3 and the Wheel graph is perfect, and it is 4 otherwise. The wheel graph is also Hamiltonian and planar.

See https://mathworld.wolfram.com/WheelGraph.html for further details.

If the optional first argument filt is present, then this should specify the category or representation the digraph being created will belong to. For example, if filt is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if filt is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument filt is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> D := WheelGraph(8);
<immutable Hamiltonian connected symmetric digraph with 8 vertices, 28\ edges>
gap> ChromaticNumber(D);
4
gap> IsHamiltonianDigraph(D);
true
```

3.5.48 WindmillGraph

▷ WindmillGraph([filt, ]n, m)

Returns: A digraph.

If $n$ and $m$ are integers greater than 1 then this operation returns the $(n, m)$-th windmill graph. The windmill graph is formed from $m$ copies of the complete graph on $n$ vertices with one shared
vertex. The graph has \( m(n - 1) + 1 \) vertices and \( m \times n \times (n - 1) / 2 \) undirected edges. The windmill graph has chromatic number \( n \) and diameter 2.

See https://mathworld.wolfram.com/WindmillGraph.html for further details.

If the optional first argument \( filt \) is present, then this should specify the category or representation the digraph being created will belong to. For example, if \( filt \) is IsMutableDigraph (3.1.2), then the digraph being created will be mutable, if \( filt \) is IsImmutableDigraph (3.1.3), then the digraph will be immutable. If the optional first argument \( filt \) is not present, then IsImmutableDigraph (3.1.3) is used by default.

Example

```gap
gap> D := WindmillGraph(4, 3);
<immutable symmetric digraph with 10 vertices, 36 edges>

gap> ChromaticNumber(D);
4
```
Chapter 4

Operators

4.1 Operators for digraphs

digraph1 = digraph2
returns true if digraph1 and digraph2 have the same vertices, and
DigraphEdges(digraph1) = DigraphEdges(digraph2), up to some re-ordering of
the edge lists.

Note that this operator does not compare the vertex labels of digraph1 and digraph2.

digraph1 < digraph2
This operator returns true if one of the following holds:

- The number \( n_1 \) of vertices in digraph1 is less than the number \( n_2 \) of vertices in digraph2;
- \( n_1 = n_2 \), and the number \( m_1 \) of edges in digraph1 is less than the number \( m_2 \) of edges in
digraph2;
- \( n_1 = n_2 \), \( m_1 = m_2 \), and DigraphEdges(digraph1) is less than
DigraphEdges(digraph2) after having both of these sets have been sorted with
respect to the lexicographical order.

4.1.1 IsSubdigraph

▷ IsSubdigraph(super, sub) (operation)

Returns: true or false.

If super and sub are digraphs, then this operation returns true if sub is a subdigraph of super,
and false if it is not.

A digraph sub is a subdigraph of a digraph super if sub and super share the same number
of vertices, and the collection of edges of super (including repeats) contains the collection of edges of
sub (including repeats).

In other words, sub is a subdigraph of super if and only if DigraphNrVertices(sub) =
DigraphNrVertices(super), and for each pair of vertices i and j, there are at least as many edges
of the form [i, j] in super as there are in sub.

Example

\[
\begin{aligned}
gap> g := \text{Digraph}([[2, 3], [1], [2, 3]]); \\
<\text{immutable digraph with 3 vertices, 5 edges}>
\end{aligned}
\]

\[
\begin{aligned}
gap> h := \text{Digraph}([[2, 3], [], [2]]);
\end{aligned}
\]
<immutable digraph with 3 vertices, 3 edges>
gap> IsSubdigraph(g, h);
true
gap> IsSubdigraph(h, g);
false
gap> IsSubdigraph(CompleteDigraph(4), CycleDigraph(4));
true
gap> IsSubdigraph(CycleDigraph(4), ChainDigraph(4));
true
gap> g := Digraph([[2, 2], [1]]);
<immutable multidigraph with 2 vertices, 3 edges>
gap> h := Digraph([[2], [1]]);
<immutable digraph with 2 vertices, 2 edges>
gap> IsSubdigraph(g, h);
true
gap> IsSubdigraph(h, g);
false

4.1.2 IsUndirectedSpanningTree
▷ IsUndirectedSpanningTree(super, sub) (operation)
▷ IsUndirectedSpanningForest(super, sub) (operation)

Returns: true or false.

The operation IsUndirectedSpanningTree returns true if the digraph sub is an undirected spanning tree of the digraph super, and the operation IsUndirectedSpanningForest returns true if the digraph sub is an undirected spanning forest of the digraph super.

An undirected spanning tree of a digraph super is a subdigraph of super that is an undirected tree (see IsSubdigraph (4.1.1) and IsUndirectedTree (6.6.9)). Note that a digraph whose MaximalSymmetricSubdigraph (3.3.5) is not connected has no undirected spanning trees (see IsConnectedDigraph (6.6.3)).

An undirected spanning forest of a digraph super is a subdigraph of super that is an undirected forest (see IsSubdigraph (4.1.1) and IsUndirectedForest (6.6.9)), and is not contained in any larger such subdigraph of super. Equivalently, an undirected spanning forest is a subdigraph of super whose connected components coincide with those of the MaximalSymmetricSubdigraph (3.3.5) of super (see DigraphConnectedComponents (5.4.9)).

Note that an undirected spanning tree is an undirected spanning forest that is connected.

gap> D := CompleteDigraph(4);
<immutable complete digraph with 4 vertices>
gap> tree := Digraph([[3], [4], [1, 4], [2, 3]]);
<immutable digraph with 4 vertices, 6 edges>
gap> IsSubdigraph(D, tree) and IsUndirectedTree(tree);
true
gap> IsUndirectedSpanningTree(D, tree);
true
gap> forest := EmptyDigraph(4);
<immutable empty digraph with 4 vertices>
gap> IsSubdigraph(D, forest) and IsUndirectedForest(forest);
true
gap> IsUndirectedSpanningForest(D, forest);
false

gap> IsSubdigraph(tree, forest);
true

gap> D := DigraphDisjointUnion(CycleDigraph(2), CycleDigraph(2));
<immutable digraph with 4 vertices, 4 edges>

gap> IsUndirectedTree(D);
false

gap> IsUndirectedForest(D) and IsUndirectedSpanningForest(D, D);
true
Chapter 5

Attributes and operations

5.1 Vertices and edges

5.1.1 DigraphVertices

\[
\text{DigraphVertices}(\text{digraph})
\]

**Returns:** A list of positive integers.

Returns the vertices of the digraph \text{digraph}.

Note that the vertices of a digraph are always the range of positive integers from 1 to the number of vertices of the graph, \text{DigraphNrVertices} (5.1.2). Arbitrary labels can be assigned to the vertices of a digraph; see \text{DigraphVertexLabels} (5.1.10) for more information about this.

Example

\[
gap> \text{gr := Digraph(}\{\text{a}, \text{b}, \text{c}\},
gap> \{\text{a}, \text{b}, \text{b}\},
gap> \{\text{b}, \text{c}, \text{a}\}\};
\]

\text{<immutable digraph with 3 vertices, 3 edges>}

\[
gap> \text{DigraphVertices(gr)};
\]

\[\{1, 2, 3\}\]

\[
gap> \text{gr := Digraph(}\{1, 2, 3, 4, 5, 7\},
gap> \{1, 2, 2, 4, 4\},
gap> \{2, 7, 5, 3, 7\}\};
\]

\text{<immutable digraph with 6 vertices, 5 edges>}

\[
gap> \text{DigraphVertices(gr)};
\]

\[\{1, 2, 3, 4, 6\}\]

\[
gap> \text{DigraphVertices(RandomDigraph(100))};
\]

\[\{1, \ldots, 100\}\]

\[
gap> D := \text{CycleDigraph(IsMutableDigraph, 3)};
\]

\text{<mutable digraph with 3 vertices, 3 edges>}

\[
gap> \text{DigraphVertices(D)};
\]

\[\{1, 2, 3\}\]

5.1.2 DigraphNrVertices

\[
\text{DigraphNrVertices}(\text{digraph})
\]

**Returns:** An non-negative integer.

Returns the number of vertices of the digraph \text{digraph}.
5.1.3 DigraphEdges

\textbf{DigraphEdges(digraph)}

\textbf{Returns:} A list of lists of two positive integers.

Returns a list of edges of the digraph \textit{digraph}, where each edge is a pair of elements of \textit{DigraphVertices} (5.1.1) of the form \([\text{source}, \text{range}]\).

The entries of \textit{DigraphEdges(digraph)} are in one-to-one correspondence with the edges of \textit{digraph}. Hence \textit{DigraphEdges(digraph)} is duplicate-free if and only if \textit{digraph} contains no multiple edges.

The entries of \textit{DigraphEdges} are guaranteed to be sorted by their first component (i.e. by the source of each edge), but they are not necessarily then sorted by the second component.

5.1.4 DigraphNrEdges

\textbf{DigraphNrEdges(digraph)}

\textbf{Returns:} An integer.

Returns the number of edges of the digraph \textit{digraph}. 
Digraphs

Example

```gap
digraph := Digraph([ [1, 3, 4, 5], [1, 2, 3, 5], [2, 4, 5], [2, 4, 5], [1] ]);;
DigraphNrEdges(digraph);
15
digraph := Digraph(["a", "b", "c"],
> ["a", "b", "b"],
> ["b", "a", "a"]);
<immutable multidigraph with 3 vertices, 3 edges>
DigraphNrEdges(digraph);
3
digraph := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
DigraphNrEdges(digraph);
3

5.1.5 DigraphNrLoops

- DigraphNrLoops(digraph)
  Returns: An integer.
  This function returns the number of loops of the digraph `digraph`. See DigraphHasLoops (6.2.1).

Example

```gap
Diagrapgh := Digraph([[2, 3], [1, 4], [3, 3, 5], [], [2, 5]]);
<immutable multidigraph with 5 vertices, 9 edges>
DigraphNrLoops(Digraph);
3
D := EmptyDigraph(5);
<immutable empty digraph with 5 vertices>
DigraphNrLoops(D);
0
D := CompleteDigraph(5);
<immutable complete digraph with 5 vertices>
DigraphNrLoops(D);
0

5.1.6 DigraphSinks

- DigraphSinks(digraph)
  Returns: A list of vertices.
  This function returns a list of the sinks of the digraph `digraph`. A sink of a digraph is a vertex
  with out-degree zero. See OutDegreeOfVertex (5.2.10).

Example

```gap
D := Digraph([ [3, 5, 2, 2], [3], [], [5, 2, 5, 3], [] ]);
<immutable multidigraph with 5 vertices, 9 edges>
DigraphSinks(Digraph);
[3, 5]
D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
DigraphSinks(Digraph);
[]
```
5.1.7 DigraphSources

▷ DigraphSources(digraph)  
Returns: A list of vertices.  
This function returns an immutable list of the sources of the digraph `digraph`. A source of a digraph is a vertex with in-degree zero. See `InDegreeOfVertex` (5.2.12).  

Example
```
gap> gr := Digraph([[3, 5, 2, 2], [3], [], [5, 2, 5, 3], []]);  
<immutable multidigraph with 5 vertices, 9 edges>
gap> DigraphSources(gr);  
[ 1, 4 ]
gap> D := CycleDigraph(IsMutableDigraph, 3);  
<mutable digraph with 3 vertices, 3 edges>
gap> DigraphSources(D);  
[ ]
```

5.1.8 DigraphTopologicalSort

▷ DigraphTopologicalSort(digraph)  
Returns: A list of positive integers, or fail.  
If `digraph` is a digraph whose only directed cycles are loops, then `DigraphTopologicalSort` returns the vertices of `digraph` ordered so that every edge’s source appears no earlier in the list than its range. If the digraph `digraph` contains directed cycles of length greater than 1, then this operation returns fail.  
See Section 1.1.1 for the definition of a directed cycle, and the definition of a loop.  
The method used for this attribute has complexity $O(m + n)$ where $m$ is the number of edges (counting multiple edges as one) and $n$ is the number of vertices in the digraph.  

Example
```
gap> D := Digraph([  
> [2, 3], [], [4, 6], [5], [], [7, 8, 9], [], [], []];  
<immutable digraph with 9 vertices, 8 edges>
gap> DigraphTopologicalSort(D);  
[ 2, 5, 4, 7, 8, 9, 6, 3, 1 ]
gap> D := Digraph(IsMutableDigraph, [[2, 3], [3], [4], []]);  
<mutable digraph with 4 vertices, 4 edges>
gap> DigraphTopologicalSort(D);  
[ 4, 3, 2, 1 ]
```

5.1.9 DigraphVertexLabel

▷ DigraphVertexLabel(digraph, i)  
▷ SetDigraphVertexLabel(digraph, i, obj)  
If `digraph` is a digraph, then the first operation returns the label of the vertex `i`. The second operation can be used to set the label of the vertex `i` in `digraph` to the arbitrary GAP object `obj`.  
The label of a vertex can be changed an arbitrary number of times. If no label has been set for the vertex `i`, then the default value is `i`.  
If `digraph` is a digraph created from a record with a component `DigraphVertices`, then the labels of the vertices are set to the value of this component.
Induced subdigraphs, and some other operations which create new digraphs from old ones, inherit their labels from their parents.

```gap
gap> D := DigraphFromDigraph6String("&DHUEe_");
<immutable digraph with 5 vertices, 11 edges>
gap> DigraphVertexLabel(D, 3);
3
gap> D := Digraph(["a", "b", "c"], [], []);
<immutable empty digraph with 3 vertices>
gap> DigraphVertexLabel(D, 2);
"b"
gap> SetDigraphVertexLabel(D, 2, "d");
<immutable empty digraph with 3 vertices>
gap> DigraphVertexLabel(D, 2);
"d"
gap> D := InducedSubdigraph(D, [1, 2]);
<immutable empty digraph with 2 vertices>
gap> InducedSubdigraph(D, [1, 2]);
<mutable empty digraph with 2 vertices>
gap> D := Digraph(IsMutableDigraph, ["e", "f", "g"], [], []);
<mutable empty digraph with 3 vertices>
gap> DigraphVertexLabel(D, 1);
"e"
gap> SetDigraphVertexLabel(D, 1, "h");
<mutable empty digraph with 3 vertices>
gap> InducedSubdigraph(D, [1, 2]);
<mutable empty digraph with 2 vertices>
gap> InducedSubdigraph(D, [1, 2]);
"h"
```

### 5.1.10 DigraphVertexLabels

- `DigraphVertexLabels(digraph)` (operation)
- `SetDigraphVertexLabels(digraph, list)` (operation)

If `digraph` is a digraph, then `DigraphVertexLabels` returns a copy of the labels of the vertices in `digraph`. `SetDigraphVertexLabels` can be used to set the labels of the vertices in `digraph` to the list of arbitrary GAP objects `list`, which must be of the same length as the number of vertices of `digraph`.

If the list `list` is immutable, then the vertex labels are set to a mutable copy of `list`. Otherwise, the labels are set to exactly `list`.

The label of a vertex can be changed an arbitrary number of times. If no label has been set for the vertex `i`, then the default value is `i`.

If `digraph` is a digraph created from a record with a component `DigraphVertices`, then the labels of the vertices are set to the value of this component. As in the above, if the component is immutable then the digraph’s vertex labels are set to a mutable copy of `DigraphVertices`. Otherwise, they are set to exactly `DigraphVertices`.

Induced subdigraphs, and other operations which create new digraphs from old ones, inherit their labels from their parents.
Digraphs

Example

```gap
D := DigraphFromDigraph6String("&DHUEe_");
<immutable digraph with 5 vertices, 11 edges>
DigraphVertexLabels(D);
[ 1 .. 5 ]
D := Digraph(["a", "b", "c"], [], []);
<immutable empty digraph with 3 vertices>
DigraphVertexLabels(D);
[ "a", "b", "c" ]
SetDigraphVertexLabel(D, 2, "d");
DigraphVertexLabels(D);
[ "a", "d", "c" ]
D := InducedSubdigraph(D, [1, 3]);
<immutable empty digraph with 2 vertices>
DigraphVertexLabels(D);
[ "a", "c" ]
D := InducedSubdigraph(D, [1, 3]);
<mutable empty digraph with 2 vertices>
SetDigraphVertexLabels(D, ["e", "f", "g"], []);
<mutable empty digraph with 3 vertices>
SetDigraphVertexLabels(D, ["h", "i", "j"]);
DigraphVertexLabels(D);
[ "h", "i", "j" ]
InducedSubdigraph(D, [1, 3]);
<mutable empty digraph with 2 vertices>
DigraphVertexLabels(D);
[ "h", "j" ]
```

5.11 DigraphEdgeLabel

- DigraphEdgeLabel(digraph, i, j)
- SetDigraphEdgeLabel(digraph, i, j, obj)

If digraph is a digraph without multiple edges, then the first operation returns the label of the edge from vertex i to vertex j. The second operation can be used to set the label of the edge between vertex i and vertex j to the arbitrary GAP object obj.

The label of an edge can be changed an arbitrary number of times. If no label has been set for the edge, then the default value is 1.

Induced subdigraphs, and some other operations which create new digraphs from old ones, inherit their edge labels from their parents. See also DigraphEdgeLabels (5.1.12).

Example

```gap
D := DigraphFromDigraph6String("&DHUEe_");
<immutable digraph with 5 vertices, 11 edges>
D := InducedSubdigraph(D, [2, 5]);
<immutable digraph with 2 vertices, 3 edges>
D := ChainDigraph(IsMutableDigraph, 5);
```
<mutable digraph with 5 vertices, 4 edges>

\[
\text{gap} \text{> } \text{DigraphEdgeLabel(D, 2, 3)}; \\
1 \\
\text{gap} \text{> } \text{SetDigraphEdgeLabel(D, 4, 5, [1729]);} \\
\text{gap} \text{> } \text{DigraphEdgeLabel(D, 4, 5)}; \\
[1729] \\
\text{gap} \text{> } \text{InducedSubdigraph(D, [4, 5]);} \\
<mutable digraph with 2 vertices, 1 edge> \\
\text{gap} \text{> } \text{DigraphEdgeLabel(D, 1, 2)}; \\
[1729] \\
\]

5.1.12 DigraphEdgeLabels

▷ DigraphEdgeLabels(digraph) (operation)
▷ SetDigraphEdgeLabels(digraph, labels) (operation)
▷ SetDigraphEdgeLabels(digraph, func) (operation)

If digraph is a digraph without multiple edges, then DigraphEdgeLabels returns a copy of the labels of the edges in digraph as a list of lists labels such that labels[i][j] is the label on the edge from vertex i to vertex OutNeighbours(digraph)[i][j]. SetDigraphEdgeLabels can be used to set the labels of the edges in digraph without multiple edges to the list labels of lists of arbitrary GAP objects such that list[i][j] is the label on the edge from vertex i to the vertex OutNeighbours(digraph>[i][j]. Alternatively SetDigraphEdgeLabels can be called with binary function func that as its second argument that when passed two vertices i and j returns the label for the edge between vertex i and vertex j.

The label of an edge can be changed an arbitrary number of times. If no label has been set for an edge, then the default value is 1.

Induced subdigraphs, and some other operations which create new digraphs from old ones, inherit their labels from their parents.

\[
\text{gap} \text{> D := DigraphFromDigraph6String("&DHUEe_"}; \\
<immutable digraph with 5 vertices, 11 edges> \\
\text{gap} \text{> } \text{DigraphEdgeLabels(D)}; \\
[ [ 1 ], [ 1, 1, 1 ], [ 1 ], [ 1, 1, 1 ], [ 1, 1, 1 ] ] \\
\text{gap} \text{> SetDigraphEdgeLabel(D, 2, 1, "d");} \\
\text{gap} \text{> } \text{DigraphEdgeLabels(D)}; \\
[ [ 1 ], [ "d", 1, 1 ], [ 1 ], [ 1, 1, 1 ], [ 1, 1, 1 ] ] \\
\text{gap} \text{> D := InducedSubdigraph(D, [1, 2, 3]);} \\
<immutable digraph with 3 vertices, 4 edges> \\
\text{gap} \text{> } \text{DigraphEdgeLabels(D)}; \\
[ [ 1 ], [ "d", 1 ], [ 1 ] ] \\
\text{gap} \text{> OutNeighbours(D);} \\
[ [ 3 ], [ 1, 3 ], [ 1 ] ] \\
\text{gap} \text{> D := CompleteBipartiteDigraph(IsMutableDigraph, 2, 3);} \\
<mutable digraph with 5 vertices, 12 edges> \\
\text{gap} \text{> } \text{DigraphEdgeLabels(D)}; \\
[ [ 1, 1, 1 ], [ 1, 1, 1 ], [ 1, 1 ], [ 1, 1 ], [ 1, 1 ] ] \\
\text{gap} \text{> SetDigraphEdgeLabel(D, 2, 4, "a");} \\
\text{gap} \text{> } \text{DigraphEdgeLabels(D)}; \\
[ [ 1, 1, 1 ], [ 1, "a", 1 ], [ 1, 1 ], [ 1, 1 ], [ 1, 1 ] ] \\
\]

Example
5.1.13 DigraphInEdges

▷ DigraphInEdges(digraph, vertex) (operation)

Returns: A list of edges.

DigraphInEdges returns the list of all edges of digraph which have vertex as their range.

Example

```gap
gap> D := Digraph([[2, 2], [3, 3], [4, 4], [1, 1]]);
<immutable multidigraph with 4 vertices, 8 edges>
gap> DigraphInEdges(D, 2);
[ [ 1, 2 ], [ 1, 2 ] ]
```

5.1.14 DigraphOutEdges

▷ DigraphOutEdges(digraph, vertex) (operation)

Returns: A list of edges.

DigraphOutEdges returns the list of all edges of digraph which have vertex as their source.

Example

```gap
gap> D := Digraph([[2, 2], [3, 3], [4, 4], [1, 1]]);
<immutable multidigraph with 4 vertices, 8 edges>
gap> DigraphOutEdges(D, 2);
[ [ 2, 3 ], [ 2, 3 ] ]
```

5.1.15 IsDigraphEdge (for digraph and list)

▷ IsDigraphEdge(digraph, list) (operation)

▷ IsDigraphEdge(digraph, u, v) (operation)

Returns: true or false.

In the first form, this function returns true if and only if the list list specifies an edge in the digraph digraph. Specifically, this operation returns true if list is a pair of positive integers where list[1] is the source and list[2] is the range of an edge in digraph, and false otherwise.

The second form simply returns true if [u, v] is an edge in digraph, and false otherwise.

Example

```gap
gap> D := Digraph([[2, 2], [6], [], [3], [], [1]]);
<immutable multidigraph with 6 vertices, 5 edges>
gap> IsDigraphEdge(D, [1, 1]);
false
gap> IsDigraphEdge(D, [1, 2]);
true
gap> IsDigraphEdge(D, [1, 8]);
false
```
5.1.16 IsMatching

▷ IsMatching(digraph, list)  
▷ IsMaximalMatching(digraph, list)  
▷ IsMaximumMatching(digraph, list)  
▷ IsPerfectMatching(digraph, list)

Returns: true or false.

If digraph is a digraph and list is a list of pairs of vertices of digraph, then IsMatching returns true if list is a matching of digraph. The operation IsMaximalMatching returns true if list is a maximal matching, IsMaximumMatching returns true if list is a maximum matching and IsPerfectMatching returns true if list is a perfect, matching of digraph, respectively. Otherwise, each of these operations return false.

A matching $M$ of a digraph $\text{digraph}$ is a subset of the edges of $\text{digraph}$, i.e. $\text{DigraphEdges(}\text{digraph})$, such that no pair of distinct edges in $M$ are incident to the same vertex of $\text{digraph}$. Note that this definition allows a matching to contain loops. See DigraphHasLoops (6.2.1). The matching $M$ is maximal if it is contained in no larger matching of the digraph, is maximum if it has the greatest cardinality among all matchings and is perfect if every vertex of the digraph is incident to an edge in the matching. Every maximum or perfect matching is maximal. Note, however, that not every perfect matching of digraphs with loops is maximum.

Example

```gap
gap> D := Digraph([[1, 2], [1, 2], [2, 3, 4], [3, 5], [1]]);
<immutable digraph with 5 vertices, 10 edges>
gap> IsMatching(D, [[2, 1], [3, 2]]);
false
gap> edges := [[3, 2]];

true

false

gap> IsPerfectMatching(D, edges);
true

gap> edges := [[1, 2], [3, 3], [4, 5]];
true

gap> IsMaximumMatching(D, edges);
false

gap> IsMaximumMatching(D, edges);
true
```

5.1.17 DigraphMaximalMatching

▷ DigraphMaximalMatching(digraph)  

Returns: A list of pairs of vertices.

This function returns a maximal matching of the digraph $\text{digraph}$. For the definition of a maximal matching, see IsMaximalMatching (5.1.16).
5.1.18 DigraphMaximumMatching

▷ DigraphMaximumMatching(digraph)  

\textbf{Returns:} A list of pairs of vertices.

This function returns a maximum matching of the digraph \texttt{digraph}.

For the definition of a maximum matching, see \texttt{IsMaximumMatching} (5.1.16). If \texttt{digraph} is bipartite (see \texttt{IsBipartiteDigraph} (6.2.3)), then the algorithm used has complexity \(O(m \cdot \sqrt{n})\). Otherwise for general graphs the complexity is \(O(m \cdot n \cdot \log(n))\). Here \(n\) is the number of vertices and \(m\) is the number of edges.

\begin{verbatim}
gap> D := DigraphFromDigraph6String("&I@EA_A?AdDp\[_c??OO");
<immutable digraph with 10 vertices, 23 edges>
gap> M := DigraphMaximumMatching(D);; IsMaximalMatching(D, M);
true
gap> Length(M);
5
\end{verbatim}

5.2 Neighbours and degree

5.2.1 AdjacencyMatrix

▷ AdjacencyMatrix(digraph)  \hspace{1cm} (attribute) 

▷ AdjacencyMatrixMutableCopy(digraph)  \hspace{1cm} (operation) 

\textbf{Returns:} A square matrix of non-negative integers.
This function returns the adjacency matrix \( \text{mat} \) of the digraph \( \text{digraph} \). The value of the matrix entry \( \text{mat}[i][j] \) is the number of edges in \( \text{digraph} \) with source \( i \) and range \( j \). If \( \text{digraph} \) has no vertices, then the empty list is returned.

The function \( \text{AdjacencyMatrix} \) returns an immutable list of lists, whereas the function \( \text{AdjacencyMatrixMutableCopy} \) returns a copy of \( \text{AdjacencyMatrix} \) that is a mutable list of mutable lists.

Example

```gap
gap> gr := Digraph(
> [2, 2, 2], [1, 3, 6, 8, 9, 10], [4, 6, 8],
> [1, 2, 3, 9], [3, 3], [3, 5, 6, 10], [1, 2, 7],
> [1, 2, 3, 10, 5, 6, 10], [1, 3, 4, 5, 8, 10],
> [2, 3, 4, 6, 7, 10]);
<immutable multidigraph with 10 vertices, 44 edges>

gap> mat := AdjacencyMatrix(gr);

gap> Display(mat);
[ [ 0, 3, 0, 0, 0, 0, 0, 0, 0, 0 ],
  [ 1, 0, 1, 0, 1, 0, 1, 1, 1, 1 ],
  [ 0, 0, 0, 1, 0, 1, 0, 1, 0, 0 ],
  [ 1, 1, 1, 0, 0, 0, 1, 0, 0, 1 ],
  [ 0, 0, 2, 0, 0, 0, 0, 0, 0, 0 ],
  [ 0, 0, 1, 0, 0, 0, 1, 0, 0, 1 ],
  [ 1, 1, 0, 0, 0, 0, 1, 0, 0, 0 ],
  [ 1, 1, 1, 0, 1, 0, 0, 0, 0, 0 ],
  [ 1, 0, 1, 1, 1, 0, 1, 0, 1, 1 ],
  [ 0, 1, 1, 1, 0, 1, 1, 0, 0, 1 ] ]

gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>

gap> Display(AdjacencyMatrix(D));
[ [ 0, 1, 0 ],
  [ 0, 0, 1 ],
  [ 1, 0, 0 ] ]
```

5.2.2 CharacteristicPolynomial

\[ \text{CharacteristicPolynomial(digraph)} \]

\textbf{Returns:} A polynomial with integer coefficients.

This function returns the characteristic polynomial of the digraph \( \text{digraph} \). That is, it returns the characteristic polynomial of the adjacency matrix of the digraph \( \text{digraph} \).

Example

```gap
gap> D := Digraph(
> [2, 2, 2], [1, 3, 6, 8, 9, 10], [4, 6, 8],
> [1, 2, 3, 9], [3, 3], [3, 5, 6, 10], [1, 2, 7],
> [1, 2, 3, 10, 5, 6, 10], [1, 3, 4, 5, 8, 10],
> [2, 3, 4, 6, 7, 10]);
<immutable multidigraph with 10 vertices, 44 edges>

gap> CharacteristicPolynomial(D);
x_1^{10}-3*x_1^9-7*x_1^8-x_1^7+14*x_1^6+x_1^5-26*x_1^4+51*x_1^3-10*x_1^2
+18*x_1-30

gap> D := CompleteDigraph(5);
<immutable complete digraph with 5 vertices>

gap> CharacteristicPolynomial(D);
```
5.2.3 BooleanAdjacencyMatrix

▷ BooleanAdjacencyMatrix(digraph) (attribute)
▷ BooleanAdjacencyMatrixMutableCopy(digraph) (operation)

Returns: A square matrix of booleans.

If digraph is a digraph with a positive number of vertices \( n \), then BooleanAdjacencyMatrix(digraph) returns the boolean adjacency matrix \( \text{mat} \) of digraph. The value of the matrix entry \( \text{mat}[j][i] \) is true if and only if there exists an edge in digraph with source \( j \) and range \( i \). If digraph has no vertices, then the empty list is returned.

Note that the boolean adjacency matrix loses information about multiple edges.

The attribute BooleanAdjacencyMatrix returns an immutable list of immutable lists, whereas the function BooleanAdjacencyMatrixMutableCopy returns a copy of the BooleanAdjacencyMatrix that is a mutable list of mutable lists.

Example

```gap
gap> gr := Digraph([[3, 4], [2, 3], [1, 2, 4], [4]]);
<immutable digraph with 4 vertices, 8 edges>
gap> PrintArray(BooleanAdjacencyMatrix(gr));
[ [ false, false, true, true ],
  [ false, true, true, false ],
  [ true, true, false, true ],
  [ false, false, true, true ] ]
gap> gr := CycleDigraph(4);
<mutable digraph with 4 vertices, 4 edges>
gap> PrintArray(BooleanAdjacencyMatrix(gr));
[ [ false, true, false, false ],
  [ false, false, true, false ],
  [ false, false, false, true ],
  [ true, false, false, false ] ]
gap> BooleanAdjacencyMatrix(EmptyDigraph(0));
[ ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> PrintArray(BooleanAdjacencyMatrix(D));
[ [ false, true, false ],
  [ false, false, true ],
  [ true, false, false ] ]
```

5.2.4 DigraphAdjacencyFunction

▷ DigraphAdjacencyFunction(digraph) (attribute)

Returns: A function.

If digraph is a digraph, then DigraphAdjacencyFunction returns a function which takes two integer parameters \( x, y \) and returns true if there exists an edge from vertex \( x \) to vertex \( y \) in digraph and false if not.
Example

```gap
gap> digraph := Digraph([[1, 2], [3], []]);
<immutable digraph with 3 vertices, 3 edges>
gap> foo := DigraphAdjacencyFunction(digraph);
function( u, v ) ... end
gap> foo(1, 1);
true
gap> foo(1, 2);
true
gap> foo(1, 3);
false
gap> foo(3, 1);
false
```

```gap
gap> gr := Digraph(["a", "b", "c"], [
    ["a", "b", "a"],
    ["b", "a", "a"]);
<immutable multidigraph with 3 vertices, 3 edges>
gap> foo := DigraphAdjacencyFunction(gr);
function( u, v ) ... end
```

```gap
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> foo := DigraphAdjacencyFunction(D);
function( u, v ) ... end
```

### 5.2.5 DigraphRange

**DigraphRange(digraph)**  
**DigraphSource(digraph)**  

**Returns:** A list of positive integers.

`DigraphRange` and `DigraphSource` return the range and source of the digraph `digraph`. More precisely, position `i` in `DigraphSource(digraph)` and `DigraphRange(digraph)` give, respectively, the source and range of the `i`th edge of `digraph`.

Example

```gap
gap> gr := Digraph([[2, 1, 3, 5], [1, 3, 4], [2, 3, [2], [1, 2, 3, 4]]]);
<immutable digraph with 5 vertices, 14 edges>
gap> DigraphSource(gr);
[ 1, 1, 1, 1, 2, 2, 2, 3, 3, 4, 5, 5, 5, 5 ]
gap> DigraphRange(gr);
[ 2, 1, 3, 5, 1, 3, 4, 2, 3, 2, 1, 2, 3, 4 ]
gap> DigraphEdges(gr);
```
[ [ 1, 2 ], [ 1, 1 ], [ 1, 3 ], [ 1, 5 ], [ 2, 1 ], [ 2, 3 ], [ 2, 4 ], [ 3, 2 ], [ 3, 3 ], [ 4, 2 ], [ 5, 1 ], [ 5, 2 ], [ 5, 3 ], [ 5, 4 ] ]

\texttt{gap} \texttt{D := CycleDigraph(IsMutableDigraph, 3);}
<mutable digraph with 3 vertices, 3 edges>
\texttt{gap} \texttt{DigraphRange(D);}
\texttt{[ 2, 3, 1 ]}
\texttt{gap} \texttt{DigraphSource(D);}
\texttt{[ 1, 2, 3 ]}

\subsection*{5.2.6 OutNeighbours}
\begin{itemize}
\item \texttt{OutNeighbours} (\texttt{digraph}) \hfill (attribute)
\item \texttt{OutNeighbors} (\texttt{digraph}) \hfill (attribute)
\item \texttt{OutNeighboursMutableCopy} (\texttt{digraph}) \hfill (operation)
\item \texttt{OutNeighborsMutableCopy} (\texttt{digraph}) \hfill (operation)
\end{itemize}

\textbf{Returns:} The adjacencies of a digraph.

\texttt{OutNeighbours} returns the list \texttt{out} of out-neighbours of each vertex of the digraph \texttt{digraph}. More specifically, a vertex \texttt{j} appears in \texttt{out}[i] each time there exists the edge \texttt{[i, j]} in \texttt{digraph}.

The function \texttt{OutNeighbours} returns an immutable list of lists, whereas the function \texttt{OutNeighboursMutableCopy} returns a copy of \texttt{OutNeighbours} which is a mutable list of mutable lists.

Note that the entries of \texttt{out} are not guaranteed to be sorted in any particular order.

\begin{example}
\begin{verbatim}
gap> gr := Digraph(["a", "b", "c"],
>               ["a", "b", "b"],
>               ["b", "a", "c"]);
<immutable digraph with 3 vertices, 3 edges>
\texttt{gap} \texttt{OutNeighbours(gr);}
\texttt{[ [ 2 ], [ 1, 3 ], [ ] ]}
\texttt{gap} \texttt{gr := Digraph([[1, 2, 3], [2, 1], [3]]);}
<immutable digraph with 3 vertices, 6 edges>
\texttt{gap} \texttt{OutNeighbours(gr);}
\texttt{[ [ 1, 2, 3 ], [ 2, 1 ], [ 3 ] ]}
\texttt{gap} \texttt{gr := DigraphByAdjacencyMatrix([}
> [ 1, 2, 1],
> [1, 1, 0],
> [0, 0, 1]]);\n<immutable multidigraph with 3 vertices, 7 edges>
\texttt{gap} \texttt{OutNeighbours(gr);}
\texttt{[ [ 1, 2, 2, 3 ], [ 1, 2 ], [ 3 ] ]}
\texttt{gap} \texttt{OutNeighboursMutableCopy(gr);}
\texttt{[ [ 1, 2, 2, 3 ], [ 1, 2 ], [ 3 ] ]}
\texttt{gap} \texttt{D := CycleDigraph(IsMutableDigraph, 3);}
<mutable digraph with 3 vertices, 3 edges>
\texttt{gap} \texttt{OutNeighbours(D);}
\texttt{[ [ 2 ], [ 3 ], [ 1 ] ]}
\end{verbatim}
\end{example}
5.2.7 InNeighbours

▷ InNeighbours(digraph)  (attribute)
▷ InNeighbors(digraph)  (attribute)
▷ InNeighboursMutableCopy(digraph)  (operation)
▷ InNeighborsMutableCopy(digraph)  (operation)

Returns: A list of lists of vertices.

InNeighbours returns the list inn of in-neighbours of each vertex of the digraph digraph. More specifically, a vertex j appears in inn[i] each time there exists an edge [j,i] in digraph.

The function InNeighbours returns an immutable list of lists, whereas the function InNeighboursMutableCopy returns a copy of InNeighbours which is a mutable list of mutable lists.

Note that the entries of inn are not necessarily sorted into ascending order, particularly if digraph was constructed via DigraphByInNeighbours (3.1.11).

Example

\begin{verbatim}
gap> gr := Digraph(\["a", "b", "c"],
   > ["a", "b", "b"],
   > ["b", "a", "c"]);
<immutable digraph with 3 vertices, 3 edges>
gap> InNeighbours(gr);
[ [ 2 ], [ 1 ], [ 2 ] ]
gap> gr := Digraph(\[[1, 2, 3], [2, 1], [3]\]);
<immutable digraph with 3 vertices, 6 edges>
gap> InNeighbours(gr);
[ [ 1, 2 ], [ 1, 2 ], [ 1, 3 ] ]
gap> gr := DigraphByAdjacencyMatrix([
   > [1, 2, 1],
   > [1, 1, 0],
   > [0, 0, 1]]);
<immutable multidigraph with 3 vertices, 7 edges>
gap> InNeighbours(gr);
[ [ 1, 2 ], [ 1, 1, 2 ], [ 1, 3 ] ]
gap> InNeighboursMutableCopy(gr);
[ [ 1, 2 ], [ 1, 1, 2 ], [ 1, 3 ] ]
gap> D := CycleDigraph(InMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> InNeighbours(D);
[ [ 3 ], [ 1 ], [ 2 ] ]
\end{verbatim}

5.2.8 OutDegrees

▷ OutDegrees(digraph)  (attribute)
▷ OutDegreeSequence(digraph)  (attribute)
▷ OutDegreeSet(digraph)  (attribute)

Returns: A list of non-negative integers.

Given a digraph digraph with n vertices, the function OutDegrees returns an immutable list out of length n, such that for a vertex i in digraph, the value of out[i] is the out-degree of vertex i. See OutDegreeOfVertex (5.2.10).

The function OutDegreeSequence returns the same list, after it has been sorted into non-increasing order.
The function `OutDegreeSet` returns the same list, sorted into increasing order with duplicate entries removed.

```
gap> D := Digraph([1, 3, 2, 2], [], [2, 1], []);  
<immutable multidigraph with 4 vertices, 6 edges>  
gap> OutDegrees(D);  
[ 4, 0, 2, 0 ]  
gap> OutDegreeSequence(D);  
[ 4, 2, 0, 0 ]  
gap> OutDegreeSet(D);  
[ 0, 2, 4 ]  
gap> D := CycleDigraph(IsMutableDigraph, 3);  
<mutable digraph with 3 vertices, 3 edges>  
gap> OutDegrees(D);  
[ 1, 1, 1 ]
```

### 5.2.9 InDegrees

- **InDegrees(digraph)**  
  - **InDegreeSequence(digraph)**  
  - **InDegreeSet(digraph)**  

**Returns:** A list of non-negative integers.

Given a digraph `digraph` with `n` vertices, the function `InDegrees` returns an immutable list `inn` of length `n`, such that for a vertex `i` in `digraph`, the value of `inn[i]` is the in-degree of vertex `i`. See `InDegreeOfVertex` (5.2.12).

The function `InDegreeSequence` returns the same list, after it has been sorted into non-increasing order.

The function `InDegreeSet` returns the same list, sorted into increasing order with duplicate entries removed.

```
gap> D := Digraph([1, 3, 2, 2, 4], [], [2, 1, 4], []);  
<immutable multidigraph with 4 vertices, 8 edges>  
gap> InDegrees(D);  
[ 2, 3, 1, 2 ]  
gap> InDegreeSequence(D);  
[ 3, 2, 2, 1 ]  
gap> InDegreeSet(D);  
[ 1, 2, 3 ]  
gap> D := CycleDigraph(IsMutableDigraph, 3);  
<mutable digraph with 3 vertices, 3 edges>  
gap> InDegrees(D);  
[ 1, 1, 1 ]
```

### 5.2.10 OutDegreeOfVertex

- **OutDegreeOfVertex(digraph, vertex)**  

**Returns:** The non-negative integer.

This operation returns the out-degree of the vertex `vertex` in the digraph `digraph`. The out-degree of `vertex` is the number of edges in `digraph` whose source is `vertex`. 
Example

```gap
D := Digraph([ [2, 2, 1], [1, 4], [2, 2, 4, 2], [1, 1, 2, 2, 1, 2, 2]]);
<immutable multidigraph with 4 vertices, 16 edges>
OutDegreeOfVertex(D, 1);
3
OutDegreeOfVertex(D, 2);
2
OutDegreeOfVertex(D, 3);
4
OutDegreeOfVertex(D, 4);
7
```

5.2.11 OutNeighboursOfVertex

▷ OutNeighboursOfVertex(digraph, vertex) (operation)
▷ OutNeighborsOfVertex(digraph, vertex) (operation)

**Returns:** A list of vertices.
This operation returns the list out of vertices of the digraph `digraph`. A vertex \(i\) appears in the list out each time there exists an edge with source `vertex` and range \(i\) in `digraph`; in particular, this means that out may contain duplicates.

Example

```gap
D := Digraph([ [2, 2, 3], [1, 3, 4], [2, 2, 3], [1, 1, 2, 2, 1, 2, 2]]);
<immutable multidigraph with 4 vertices, 16 edges>
OutNeighboursOfVertex(D, 1);
[2, 2, 3]
OutNeighboursOfVertex(D, 3);
[2, 2, 3]
```

5.2.12 InDegreeOfVertex

▷ InDegreeOfVertex(digraph, vertex) (operation)

**Returns:** A non-negative integer.
This operation returns the in-degree of the vertex `vertex` in the digraph `digraph`. The in-degree of `vertex` is the number of edges in `digraph` whose range is `vertex`.

Example

```gap
D := Digraph([ [2, 2, 1], [1, 4], [2, 2, 4, 2], [1, 1, 2, 2, 1, 2, 2]]);
<immutable multidigraph with 4 vertices, 16 edges>
InDegreeOfVertex(D, 1);
5
InDegreeOfVertex(D, 2);
9
InDegreeOfVertex(D, 3);
0
InDegreeOfVertex(D, 4);
2
```
5.2.13  InNeighboursOfVertex

▷ InNeighboursOfVertex(digraph, vertex)  (operation)
▷ InNeighborsOfVertex(digraph, vertex)  (operation)

Returns: A list of positive vertices.
This operation returns the list \( \text{inn} \) of vertices of the digraph \( \text{digraph} \). A vertex \( i \) appears in the list \( \text{inn} \) each time there exists an edge with source \( i \) and range \( \text{vertex} \) in \( \text{digraph} \); in particular, this means that \( \text{inn} \) may contain duplicates.

Example

\[
\text{gap> D := Digraph([}
> [2, 2, 3], [1, 3, 4], [2, 2, 3], [1, 1, 2, 2, 1, 2, 2]));
\text{<immutable multidigraph with 4 vertices, 16 edges>}
\text{gap> InNeighboursOfVertex(D, 1):}
[2, 4, 4, 4]
\text{gap> InNeighborsOfVertex(D, 2):}
[1, 1, 3, 3, 4, 4, 4, 4]
\]

5.2.14  DigraphLoops

▷ DigraphLoops(digraph)  (attribute)

Returns: A list of vertices.
If \( \text{digraph} \) is a digraph, then \text{DigraphLoops} returns the list consisting of the \text{DigraphVertices (5.1.1)} of \( \text{digraph} \) at which there is a loop. See \text{DigraphHasLoops (6.2.1)}.

Example

\[
\text{gap> D := Digraph([}
> [2], [3], []);
\text{<immutable digraph with 3 vertices, 2 edges>}
\text{gap> DigraphHasLoops(D):}
\text{false}
\text{gap> DigraphLoops(D):}
[ ]
\text{gap> D := Digraph([[3, 5], [1], [2, 4, 3], [4], [2, 1]]);}
\text{<immutable digraph with 5 vertices, 9 edges>}
\text{gap> DigraphLoops(D):}
[3, 4]
\text{gap> D := Digraph(IsMutableDigraph, [[1], [1]]);}
\text{<mutable digraph with 2 vertices, 2 edges>}
\text{gap> DigraphLoops(D):}
[1]
\]

5.2.15  DegreeMatrix

▷ DegreeMatrix(digraph)  (attribute)

Returns: A square matrix of non-negative integers.
Returns the out-degree matrix \( \text{mat} \) of the digraph \( \text{digraph} \). The value of the \( i \)th diagonal matrix entry is the out-degree of the vertex \( i \) in \( \text{digraph} \). All off-diagonal entries are 0. If \( \text{digraph} \) has no vertices, then the empty list is returned.
See \text{OutDegrees (5.2.8)} for more information.

Example

\[
\text{gap> D := Digraph([[1, 2, 3], [4], [1, 3, 4], []]);}
\text{<immutable digraph with 4 vertices, 7 edges>}
\]
Digraphs

gap> PrintArray(DegreeMatrix(D));
[ [ 3, 0, 0, 0 ],
  [ 0, 1, 0, 0 ],
  [ 0, 0, 3, 0 ],
  [ 0, 0, 0, 0 ] ]
gap> D := CycleDigraph(5);;
gap> PrintArray(DegreeMatrix(D));
[ [ 1, 0, 0, 0, 0 ],
  [ 0, 1, 0, 0, 0 ],
  [ 0, 0, 1, 0, 0 ],
  [ 0, 0, 0, 1, 0 ],
  [ 0, 0, 0, 0, 1 ] ]
gap> DegreeMatrix(EmptyDigraph(0));
[ ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> Display(DegreeMatrix(D));
[ [ 1, 0, 0 ],
  [ 0, 1, 0 ],
  [ 0, 0, 1 ] ]

5.2.16 LaplacianMatrix

▷ LaplacianMatrix(digraph) (attribute)

Returns: A square matrix of integers.

Returns the out-degree Laplacian matrix mat of the digraph digraph. The out-degree Laplacian matrix is defined as DegreeMatrix(digraph) - AdjacencyMatrix(digraph). If digraph has no vertices, then the empty list is returned.

See DegreeMatrix (5.2.15) and AdjacencyMatrix (5.2.1) for more information.

Example
gap> gr := Digraph([[1, 2, 3], [4], [1, 3, 4], []]);
<immutable digraph with 4 vertices, 7 edges>
gap> PrintArray(LaplacianMatrix(gr));
[ [ 2, -1, -1, 0 ],
  [ 0, 1, 0, -1 ],
  [ -1, 0, 2, -1 ],
  [ 0, 0, 0, 0 ] ]
gap> LaplacianMatrix(EmptyDigraph(0));
[ ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> Display(LaplacianMatrix(D));
[ [ 1, -1, 0 ],
  [ 0, 1, -1 ],
  [ -1, 0, 1 ] ]
5.3 Orders

5.3.1 PartialOrderDigraphMeetOfVertices

\begin{itemize}
\item \texttt{PartialOrderDigraphMeetOfVertices(digraph, u, v)} \hspace{1cm} \text{(operation)}
\item \texttt{PartialOrderDigraphJoinOfVertices(digraph, u, v)} \hspace{1cm} \text{(operation)}
\end{itemize}

\textbf{Returns:} A positive integer or fail

If the first argument is a partial order digraph IsPartialOrderDigraph (6.4.2) then these operations return the meet, or the join, of the two input vertices. If the meet (or join) is does not exist then fail is returned. The meet (or join) is guaranteed to exist when the first argument satisfies IsMeetSemilatticeDigraph (6.4.3) or IsJoinSemilatticeDigraph (6.4.3) - see the documentation for these properties for the definition of the meet (or the join).

\begin{verbatim}
gap> D := Digraph([[1], [1, 2], [1, 3], [1, 2, 3, 4]]);<immutable digraph with 4 vertices, 9 edges>
gap> PartialOrderDigraphMeetOfVertices(D, 2, 3); 4
\end{verbatim}

\begin{verbatim}
gap> D := Digraph([[1], [1, 2], [1, 3], [1, 2, 4]]);
<immutable digraph with 4 vertices, 8 edges>
gap> PartialOrderDigraphMeetOfVertices(D, 3, 4); fail
\end{verbatim}

\begin{verbatim}
gap> D := Digraph([[1], [2], [1, 2, 3], [1, 2, 4]]);
<immutable digraph with 4 vertices, 8 edges>
gap> PartialOrderDigraphJoinOfVertices(D, 1, 2); fail
\end{verbatim}

\begin{verbatim}
gap> D := Digraph([[1], [2], [1, 2, 3], [1, 2, 4]]);
<immutable digraph with 4 vertices, 8 edges>
gap> PartialOrderDigraphJoinOfVertices(D, 1, 2); fail
\end{verbatim}

5.3.2 NonUpperSemimodularPair

\begin{itemize}
\item \texttt{NonUpperSemimodularPair(D)} \hspace{1cm} \text{(attribute)}
\item \texttt{NonLowerSemimodularPair(D)} \hspace{1cm} \text{(attribute)}
\end{itemize}

\textbf{Returns:} A pair of vertices or fail.

NonUpperSemimodularPair returns a pair of vertices in the digraph \( D \) that witnesses the fact that \( D \) does not represent an upper semimodular lattice, if such a pair exists.

If the digraph \( D \) does not satisfy IsLatticeDigraph (6.4.3), then an error is given. Otherwise if the digraph \( D \) does satisfy IsLatticeDigraph (6.4.3), then either a non-upper semimodular pair of vertices is returned, or fail is returned if no such pair exists (meaning that \( D \) is an upper semimodular lattice).

NonLowerSemimodularPair behaves in the analogous way to NonUpperSemimodularPair with respect to lower semimodularity.

See IsUpperSemimodularDigraph (6.4.5) and IsLowerSemimodularDigraph (6.4.5) for the definition of upper semimodularity of a lattice.
5.4 Reachability and connectivity

5.4.1 DigraphDiameter

▷ DigraphDiameter(digraph) (attribute)

Returns: An integer or fail.

This function returns the diameter of the digraph digraph.

If a digraph digraph is strongly connected and has at least 1 vertex, then the diameter is the maximum shortest distance between any pair of distinct vertices. Otherwise then the diameter of digraph is undefined, and this function returns the value fail.

See DigraphShortestDistances (5.4.3).

5.4.2 DigraphShortestDistance (for a digraph and two vertices)

▷ DigraphShortestDistance(digraph, u, v) (operation)

▷ DigraphShortestDistance(digraph, list) (operation)

▷ DigraphShortestDistance(digraph, list1, list2) (operation)

Returns: An integer or fail

If there is a directed path in the digraph digraph between vertex u and vertex v, then this operation returns the length of the shortest such directed path. If no such directed path exists, then this operation returns fail. See Section 1.1.1 for the definition of a directed path.

If the second form is used, then list should be a list of length two, containing two positive integers which correspond to the vertices u and v.

Note that as usual, a vertex is considered to be at distance 0 from itself.
If the third form is used, then \textit{list1} and \textit{list2} are both lists of vertices. The shortest directed path between \textit{list1} and \textit{list2} is then the length of the shortest directed path which starts with a vertex in \textit{list1} and terminates at a vertex in \textit{list2}, if such directed path exists. If \textit{list1} and \textit{list2} have non-empty intersection, the operation returns 0.

\begin{verbatim}
Example

gap> D := Digraph([[2], [3], [1, 4], [1, 3], [5]]);
<immutable digraph with 5 vertices, 7 edges>
gap> DigraphShortestDistance(D, 1, 3);
2
gap> DigraphShortestDistance(D, [3, 3]);
0
gap> DigraphShortestDistance(D, 5, 2);
fail
gap> DigraphShortestDistance(D, [1, 2], [4, 5]);
2
gap> DigraphShortestDistance(D, [1, 3], [3, 5]);
0
\end{verbatim}

\section*{5.4.3 DigraphShortestDistances}

\begin{verbatim}
▷ DigraphShortestDistances(digraph) (attribute)

Returns: A square matrix.

If \textit{digraph} is a digraph with \( n \) vertices, then this function returns an \( n \times n \) matrix \textit{mat}, where each entry is either a non-negative integer, or \textit{fail}. If \( n = 0 \), then an empty list is returned.

If there is a directed path from vertex \( i \) to vertex \( j \), then the value of \textit{mat}[i][j] is the length of the shortest such directed path. If no such directed path exists, then the value of \textit{mat}[i][j] is \textit{fail}. We use the convention that the distance from every vertex to itself is 0, i.e. \textit{mat}[i][i] = 0 for all vertices \( i \).

The method used in this function is a version of the Floyd-Warshall algorithm, and has complexity \( O(n^3) \).

Example

\begin{verbatim}
gap> D := Digraph([[1, 2], [3], [1, 2], [4]]);
<immutable digraph with 4 vertices, 6 edges>
gap> mat := DigraphShortestDistances(D);

[ [ 0, 1, 2, fail ],
  [ 2, 0, 1, fail ],
  [ 1, 1, 0, fail ],
  [ fail, fail, fail, 0 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 3);
<mutable digraph with 3 vertices, 3 edges>
gap> DigraphShortestDistances(D);

[ [ 0, 1, 2 ], [ 2, 0, 1 ], [ 1, 2, 0 ] ]
\end{verbatim}

\section*{5.4.4 DigraphLongestDistanceFromVertex}

\begin{verbatim}
▷ DigraphLongestDistanceFromVertex(digraph, v) (operation)

Returns: An integer, or infinity.
\end{verbatim}
If digraph is a digraph and v is a vertex in digraph, then this operation returns the length of the longest directed walk in digraph which begins at vertex v. See Section 1.1.1 for the definitions of directed walk, directed cycle, and loop.

- If there exists a directed walk starting at vertex v which traverses a loop or a directed cycle, then we consider there to be a walk of infinite length from v (realised by repeatedly traversing the loop/directed cycle), and so the result is infinity. To disallow walks using loops, try using DigraphRemoveLoops (3.3.25):
  DigraphLongestDistanceFromVertex(DigraphRemoveLoops(digraph, v)).

- Otherwise, if all directed walks starting at vertex v have finite length, then the length of the longest such walk is returned.

Note that the result is 0 if and only if v is a sink of digraph. See DigraphSinks (5.1.6).

Example

```
gap> D := Digraph([[2], [3, 4], [1], [5], [1], [6]]);
<immutable digraph with 6 vertices, 5 edges>
gap> DigraphLongestDistanceFromVertex(D, 1);
3
gap> DigraphLongestDistanceFromVertex(D, 3);
0
gap> 3 in DigraphSinks(D);
true
```

5.4.5 DigraphDistanceSet (for a digraph, a pos int, and an int)

▷ DigraphDistanceSet(digraph, vertex, distance)  
▷ DigraphDistanceSet(digraph, vertex, distances)  

Returns: A list of vertices

This operation returns the list of all vertices in digraph digraph such that the shortest distance to a vertex vertex is distance or is in the list distances.

digraph should be a digraph, vertex should be a positive integer, distance should be a non-negative integer, and distances should be a list of non-negative integers.

Example

```
gap> D := Digraph([[2], [3], [1, 4], [1, 3]]);
<immutable digraph with 4 vertices, 6 edges>
gap> DigraphDistanceSet(D, 2, [1, 2]);
[ 3, 1, 4 ]
gap> DigraphDistanceSet(D, 3, 1);
[ 1, 4 ]
gap> DigraphDistanceSet(D, 2, 0);
[ 2 ]
```

5.4.6 DigraphGirth

▷ DigraphGirth(digraph)  

Returns: An integer, or infinity.
This attribute returns the \textit{girth} of the digraph \texttt{digraph}. The \textit{girth} of a digraph is the length of its shortest simple circuit. See Section 1.1.1 for the definitions of simple circuit, directed cycle, and loop.

If \texttt{digraph} has no directed cycles, then this function will return \texttt{infinity}. If \texttt{digraph} contains a loop, then this function will return 1.

In the worst case, the method used in this function is a version of the Floyd-Warshall algorithm, and has complexity $O(n^3)$, where $n$ is the number of vertices in \texttt{digraph}. If the digraph has known automorphisms [see \texttt{DigraphGroup (7.2.10)}], then the performance is likely to be better.

For symmetric digraphs, see also \texttt{DigraphUndirectedGirth (5.4.8)}.

\begin{verbatim}
Example

gap> D := Digraph([[1], [1]]);
<immutable digraph with 2 vertices, 2 edges>
gap> DigraphGirth(D);
1

gap> D := Digraph([[2, 3], [3], [4], [1]]);
<immutable digraph with 4 vertices, 4 edges>
gap> DigraphGirth(D);
infinity

gap> D := Digraph([[2, 3], [3], [4], [1]]);
<immutable digraph with 4 vertices, 5 edges>
gap> DigraphGirth(D);
3

gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 6, 2);
<mutable digraph with 12 vertices, 36 edges>
gap> DigraphGirth(D);
2
\end{verbatim}

\section{DigraphOddGirth}

\noindent \texttt{DigraphOddGirth(digraph)} \hfill (attribute)

\textbf{Returns:} An integer, or \texttt{infinity}.

This attribute returns the \textit{odd girth} of the digraph \texttt{digraph}. The \textit{odd girth} of a digraph is the length of its shortest simple circuit of odd length. See Section 1.1.1 for the definitions of simple circuit, directed cycle, and loop.

If \texttt{digraph} has no directed cycles of odd length, then this function will return \texttt{infinity}, even if \texttt{digraph} has a directed cycle of even length. If \texttt{digraph} contains a loop, then this function will return 1.

See also \texttt{DigraphGirth (5.4.6)}.

\begin{verbatim}
Example

gap> D := Digraph([[2], [3, 1], [1]]);
<immutable digraph with 3 vertices, 4 edges>
gap> DigraphOddGirth(D);
3

gap> D := CycleDigraph(4);
<immutable cycle digraph with 4 vertices>
gap> DigraphOddGirth(D);
infinity

gap> D := Digraph([[2], [3], [], [3], [4]]);
<immutable digraph with 5 vertices, 4 edges>
gap> DigraphOddGirth(D);
infinity
\end{verbatim}
Digraphs

5.4.8 DigraphUndirectedGirth

▷ DigraphUndirectedGirth(digraph)

Returns: An integer or infinity.

If digraph is a symmetric digraph, then this function returns the girth of digraph when treated as an undirected graph (i.e. each pair of edges \([i,j]\) and \([j,i]\) is treated as a single edge between \(i\) and \(j\)).

The girth of an undirected graph is the length of its shortest simple cycle, i.e. the shortest non-trivial path starting and ending at the same vertex and passing through no vertex or edge more than once.

If digraph has no cycles, then this function will return infinity.

Example

```gap
gap> D := Digraph([[2, 4], [1, 3], [2, 4], [1, 3]]);
<immutable digraph with 4 vertices, 8 edges>
gap> DigraphUndirectedGirth(D);
4
gap> D := Digraph([[2], [1, 3], [2]]);
<immutable digraph with 3 vertices, 4 edges>
gap> DigraphUndirectedGirth(D);
infinity
gap> D := Digraph([[1], [], [4], [3]]);
<immutable digraph with 4 vertices, 3 edges>
gap> DigraphUndirectedGirth(D);
1
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 9, 2);
<mutable digraph with 18 vertices, 54 edges>
gap> DigraphUndirectedGirth(D);
5
```

5.4.9 DigraphConnectedComponents

▷ DigraphConnectedComponents(digraph)  

▷ DigraphNrConnectedComponents(digraph)

Returns: A record.

This function returns the record wcc corresponding to the weakly connected components of the digraph digraph. Two vertices of digraph are in the same weakly connected component whenever they are equal, or there exists a directed path (ignoring the orientation of edges) between them. More formally, two vertices are in the same weakly connected component of digraph if and only if they are in the same strongly connected component (see DigraphStronglyConnectedComponents (5.4.11)) of the DigraphSymmetricClosure (3.3.12) of digraph.

The set of weakly connected components is a partition of the vertex set of digraph.
The record \texttt{wcc} has 2 components: \texttt{comps} and \texttt{id}. The component \texttt{comps} is a list of the weakly connected components of \texttt{digraph} (each of which is a list of vertices). The component \texttt{id} is a list such that the vertex \texttt{i} is an element of the weakly connected component \texttt{comps}[\texttt{id}[i]].

The method used in this function has complexity $O(m+n)$, where $m$ is the number of edges and $n$ is the number of vertices in the digraph.

\texttt{DigraphNrConnectedComponents(digraph)} is simply a shortcut for $\text{Length(DigraphConnectedComponents(digraph).comps)}$, and is no more efficient.

\begin{verbatim}
gap> gr := Digraph([[2], [3, 1], []]);
<immutable digraph with 3 vertices, 3 edges>
gap> DigraphConnectedComponents(gr);
rec( comps := [ [ 1, 2, 3 ] ], id := [ 1, 1, 1 ] )
gap> gr := Digraph([[1], [1, 2], []]);
<immutable digraph with 3 vertices, 3 edges>
gap> DigraphConnectedComponents(gr);
rec( comps := [ [ 1, 2 ], [ 3 ] ], id := [ 1, 1, 2 ] )
gap> DigraphNrConnectedComponents(gr);
2
gap> gr := EmptyDigraph(0);
<immutable empty digraph with 0 vertices>
gap> DigraphConnectedComponents(gr);
rec( comps := [ ], id := [ ] )
gap> D := CycleDigraph(IsMutableDigraph, 2);
<mutable digraph with 2 vertices, 2 edges>
gap> G := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
gap> DigraphDisjointUnion(D, G);
<mutable digraph with 5 vertices, 5 edges>
gap> DigraphConnectedComponents(D);
rec( comps := [ [ 1, 2 ], [ 3, 4, 5 ] ], id := [ 1, 1, 2, 2, 2 ] )
\end{verbatim}

\subsection*{5.4.10 DigraphConnectedComponent}

\begin{verbatim}
\texttt{DigraphConnectedComponent(digraph, vertex)}
\end{verbatim}

\textbf{Returns:} A list of vertices.

If \texttt{vertex} is a vertex in the digraph \texttt{digraph}, then this operation returns the connected component of \texttt{vertex} in \texttt{digraph}. See \texttt{DigraphConnectedComponents (5.4.9)} for more information.

\begin{verbatim}
\texttt{D := Digraph([[3], [2], [1, 2], [4]]);}
<immutable digraph with 4 vertices, 5 edges>
\texttt{DigraphConnectedComponent(D, 3);}
[ 1, 2, 3 ]
\texttt{DigraphConnectedComponent(D, 2);}
[ 1, 2, 3 ]
\texttt{DigraphConnectedComponent(D, 4);}
[ 4 ]
\end{verbatim}

\subsection*{5.4.11 DigraphStronglyConnectedComponents}

\begin{verbatim}
\texttt{DigraphStronglyConnectedComponents(digraph)}
\end{verbatim}

\begin{verbatim}
\texttt{DigraphNrStronglyConnectedComponents(digraph)}
\end{verbatim}
**Returns:** A record.

This function returns the record \texttt{scc} corresponding to the strongly connected components of the digraph \texttt{digraph}. Two vertices of \texttt{digraph} are in the same strongly connected component whenever they are equal, or there is a directed path from each vertex to the other. The set of strongly connected components is a partition of the vertex set of \texttt{digraph}.

The record \texttt{scc} has 2 components: \texttt{comps} and \texttt{id}. The component \texttt{comps} is a list of the strongly connected components of \texttt{digraph} (each of which is a list of vertices). The component \texttt{id} is a list such that the vertex \texttt{i} is an element of the strongly connected component \texttt{comps}[\texttt{id}[\texttt{i}]].

The method used in this function is a non-recursive version of Gabow’s Algorithm [Gab00] and has complexity \(O(m + n)\) where \(m\) is the number of edges (counting multiple edges as one) and \(n\) is the number of vertices in the digraph.

\texttt{DigraphNrStronglyConnectedComponents(digraph)} is simply a shortcut for \(\text{Length}(\text{DigraphStronglyConnectedComponents(digraph)})\text{.comps})\), and is no more efficient.

### Example

```gap
gap> gr := Digraph([[2], [3, 1], [ ]]);
<immutable digraph with 3 vertices, 3 edges>
gap> DigraphStronglyConnectedComponents(gr);
rec( comps := [ [ 3 ], [ 1, 2 ] ], id := [ 2, 2, 1 ] )
gap> DigraphNrStronglyConnectedComponents(gr);
2
gap> D := DigraphDisjointUnion(CycleDigraph(4), CycleDigraph(5));
<immutable digraph with 9 vertices, 9 edges>
gap> DigraphStronglyConnectedComponents(D);
rec( comps := [ [ 1, 2, 3, 4 ], [ 5, 6, 7, 8, 9 ] ],
    id := [ 1, 1, 1, 1, 2, 2, 2, 2, 2 ] )
gap> DigraphNrStronglyConnectedComponents(D);
2
gap> D := CycleDigraph(IsMutableDigraph, 2);
<mutable digraph with 2 vertices, 2 edges>
gap> G := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
gap> DigraphDisjointUnion(D, G);
<mutable digraph with 5 vertices, 5 edges>
gap> DigraphStronglyConnectedComponents(D);
rec( comps := [ [ 1, 2 ], [ 3, 4, 5 ] ], id := [ 1, 1, 2, 2, 2 ] )
```

### 5.4.12 DigraphStronglyConnectedComponent

\texttt{DigraphStronglyConnectedComponent(digraph, vertex)}

**Returns:** A list of vertices.

If \texttt{vertex} is a vertex in the digraph \texttt{digraph}, then this operation returns the strongly connected component of \texttt{vertex} in \texttt{digraph}. See \texttt{DigraphStronglyConnectedComponents (5.4.11)} for more information.

### Example

```gap
gap> D := Digraph([[3], [2], [1, 2], [3]]);
<immutable digraph with 4 vertices, 5 edges>
gap> DigraphStronglyConnectedComponent(D, 3);
[ 1, 3 ]
gap> DigraphStronglyConnectedComponent(D, 2);
```
5.4.13 DigraphBicomponents

▷ DigraphBicomponents(digraph)  
Returns: A pair of lists of vertices, or fail.

If digraph is a bipartite digraph, i.e. if it satisfies IsBipartiteDigraph (6.2.3), then DigraphBicomponents returns a pair of bicomponents of digraph. Otherwise, DigraphBicomponents returns fail.

For a bipartite digraph, the vertices can be partitioned into two non-empty sets such that the source and range of any edge are in distinct sets. The parts of this partition are called bicomponents of digraph. Equivalently, a pair of bicomponents of digraph consists of the color-classes of a 2-coloring of digraph.

For a bipartite digraph with at least 3 vertices, there is a unique pair of bicomponents of bipartite if and only if the digraph is connected. See IsConnectedDigraph (6.6.3) for more information.

Example

\[
\begin{align*}
gap> D := \text{CycleDigraph}(3); \\
&<\text{immutable cycle digraph with 3 vertices}> \\
gap> \text{DigraphBicomponents}(D); \\
&\text{fail} \\
gap> D := \text{ChainDigraph}(5); \\
&<\text{immutable chain digraph with 5 vertices}> \\
gap> \text{DigraphBicomponents}(D); \\
&[ [ 1, 3, 5 ], [ 2, 4 ] ] \\
gap> D := \text{Digraph}([[5], [1, 4], [5], [5], []]); \\
&<\text{immutable digraph with 5 vertices, 5 edges}> \\
gap> \text{DigraphBicomponents}(D); \\
&[ [ 1, 3, 4 ], [ 2, 5 ] ] \\
gap> D := \text{CompleteBipartiteDigraph}(\text{IsMutableDigraph}, 2, 3); \\
&<\text{mutable digraph with 5 vertices, 12 edges}> \\
gap> \text{DigraphBicomponents}(D); \\
&[ [ 1, 2 ], [ 3, 4, 5 ] ]
\end{align*}
\]

5.4.14 ArticulationPoints

▷ ArticulationPoints(digraph)  
Returns: A list of vertices.

A connected digraph is biconnected if it is still connected (in the sense of IsConnectedDigraph (6.6.3)) when any vertex is removed. If the digraph digraph is not biconnected but is connected, then any vertex v of digraph whose removal makes the resulting digraph disconnected is called an articulation point.

ArticulationPoints returns a list of the articulation points of digraph, if any, and, in particular, returns the empty list if digraph is not connected.

Multiple edges are ignored by this method.

The method used in this operation has complexity $O(m + n)$ where $m$ is the number of edges and $n$ is the number of vertices in the digraph.
If \( D \) has a bridge (see Bridges (5.4.15)), then a node incident to the bridge is an articulation point if and only if it has degree at least 2. It follows that if \( D \) has a bridge and at least 3 nodes, then at least one of the nodes incident to the bridge is an articulation point. The converse does not hold, there are digraphs with articulation points, but no bridges.

See also IsBiconnectedDigraph (6.6.4) and IsBridgelessDigraph (6.6.5).

Example

\[
\begin{array}{l}
gap> \text{ArticulationPoints(CycleDigraph(5));} \\
[ \ ] \\
gap> \text{D := Digraph([[2, 7], [3, 5], [4], [2], [6], [1], []]);} \\
gap> \text{ArticulationPoints(D);} \\
[ 2, 1 ] \\
gap> \text{ArticulationPoints(ChainDigraph(5));} \\
[ 4, 3, 2 ] \\
gap> \text{ArticulationPoints(NullDigraph(5));} \\
[ ] \\
gap> \text{D := ChainDigraph(IsMutableDigraph, 4);} \\
<\text{mutable digraph with 4 vertices, 3 edges}> \\
gap> \text{ArticulationPoints(D);} \\
[ 3, 2 ]
\end{array}
\]

5.4.15 Bridges

\( \text{Bridges}(D) \) (attribute)

Returns: A (possibly empty) list of edges.

A connected digraph is 2-edge-connected if it is still connected (in the sense of IsConnectedDigraph (6.6.3)) when any edge is removed. If the digraph \( D \) is not 2-edge-connected but is connected, then any edge \([u, v]\) of \( D \) whose removal makes the resulting digraph disconnected is called a bridge.

Bridges returns a list of the bridges of \( D \), if any, and, in particular, returns the empty list if \( D \) is not connected.

Multiple edges are ignored by this method.

The method used in this operation has complexity \( O(m + n) \) where \( m \) is the number of edges and \( n \) is the number of vertices in the digraph.

If \( D \) has a bridge, then a node incident to the bridge is an articulation point (see ArticulationPoints (5.4.14)) if and only if it has degree at least 2. It follows that if \( D \) has a bridge and at least 3 nodes, then at least one of the nodes incident to the bridge is an articulation point. The converse does not hold, there are digraphs with articulation points, but no bridges.

See also IsBiconnectedDigraph (6.6.4) and IsBridgelessDigraph (6.6.5).

Example

\[
\begin{array}{l}
gap> \text{D := Digraph([[2, 5], [1, 3, 4, 5], [2, 4], [2, 3], [1, 2]]);} \\
<\text{immutable digraph with 5 vertices, 12 edges}> \\
gap> \text{Bridges(D);} \\
[ ] \\
gap> \text{D := Digraph([[2], [3], [4], [2]]);} \\
<\text{immutable digraph with 4 vertices, 4 edges}> \\
gap> \text{Bridges(D);} \\
[ [ 1, 2 ] ] \\
gap> \text{Bridges(ChainDigraph(2));} \\
[ [ 1, 2 ] ]
\end{array}
\]
5.4.16 StrongOrientation

▷ StrongOrientation(D) (operation)
▷ StrongOrientationAttr(D) (attribute)

Returns: A digraph or fail.

A strong orientation of a connected symmetric digraph \( D \) (if it exists) is a strongly connected subdigraph \( C \) of \( D \) such that for every edge \([u, v]\) of \( D \) either \([u, v]\) or \([v, u]\) is an edge of \( C \) but not both. Robbin’s Theorem states that a digraph admits a strong orientation if and only if it is bridgeless (see IsBridgelessDigraph (6.6.5)).

This operation returns a strong orientation of the digraph \( D \) if \( D \) is symmetric and \( D \) admits a strong orientation. If \( D \) is symmetric but does not admit a strong orientation, then fail is returned. If \( D \) is not symmetric, then an error is given.

If \( D \) is immutable, StrongOrientation(D) returns an immutable digraph, and if \( D \) is mutable, then StrongOrientation(D) returns a mutable digraph.

The method used in this operation has complexity \( O(m + n) \) where \( m \) is the number of edges and \( n \) is the number of vertices in the digraph.

Example

\[
\text{gap> StrongOrientation(DigraphSymmetricClosure(CycleDigraph(5)))} = \text{CycleDigraph(5)}; \\
\text{true} \\
\text{gap> D := DigraphSymmetricClosure(Digraph([2, 7], [3, 5], [4], [2], [6], [1], [1]));} \\
\text{false} \\
\text{gap> StrongOrientation(D); fail} \\
\text{gap> IsBridgelessDigraph(D);} \\
<\text{immutable empty digraph with 0 vertices}> \\
\text{gap> StrongOrientation(NullDigraph(0)); fail} \\
\text{gap> StrongOrientation(DigraphDisjointUnion(CompleteDigraph(3), CompleteDigraph(3))); fail}
\]

5.4.17 DigraphPeriod

▷ DigraphPeriod(digraph) (attribute)

Returns: An integer.

This function returns the period of the digraph \( \text{digraph} \).

If a digraph \( \text{digraph} \) has at least one directed cycle, then the period is the greatest positive integer which divides the lengths of all directed cycles of \( \text{digraph} \). If \( \text{digraph} \) has no directed cycles, then this function returns 0. See Section 1.1.1 for the definition of a directed cycle.

A digraph with a period of 1 is said to be aperiodic. See IsAperiodicDigraph (6.6.7).

Example

\[
\text{gap> D := Digraph([[6], [1], [2], [3], [4, 4], [5]]);} \\
<\text{immutable multidigraph with 6 vertices, 7 edges}> \\
\text{gap> DigraphPeriod(D);}
6

\begin{verbatim}
gap> D := Digraph([[2], [3, 5], [4], [5], [1, 2]]);
<immutable digraph with 5 vertices, 7 edges>
gap> DigraphPeriod(D);
1
gap> D := ChainDigraph(2);
<immutable chain digraph with 2 vertices>
gap> DigraphPeriod(D);
0
gap> IsAcyclicDigraph(D);
true
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 5, 2);
<mutable digraph with 10 vertices, 30 edges>
gap> DigraphPeriod(D);
1
\end{verbatim}

5.4.18 DigraphFloydWarshall

\begin{verbatim}
DigraphFloydWarshall(digraph, func, nopath, edge)
\end{verbatim}

\textbf{Returns:} A matrix.

If \textit{digraph} is a digraph with \( n \) vertices, then this operation returns an \( n \times n \) matrix \textit{mat} containing the output of a generalised version of the Floyd-Warshall algorithm, applied to \textit{digraph}.

The operation \textit{DigraphFloydWarshall} is customised by the arguments \textit{func}, \textit{nopath}, and \textit{edge}. The arguments \textit{nopath} and \textit{edge} can be arbitrary \textit{GAP} objects. The argument \textit{func} must be a function which accepts 4 arguments: the matrix \textit{mat}, followed by 3 positive integers. The function \textit{func} is where the work to calculate the desired outcome must be performed.

This method initialises the matrix \textit{mat} by setting entry \textit{mat}[i][j] to equal \textit{edge} if there is an edge with source \( i \) and range \( j \), and by setting entry \textit{mat}[i][j] to equal \textit{nopath} otherwise. The final part of \textit{DigraphFloydWarshall} then calls the function \textit{func} inside three nested for loops, over the vertices of \textit{digraph}:

\begin{verbatim}
for i in DigraphsVertices(digraph) do
    for j in DigraphsVertices(digraph) do
        for k in DigraphsVertices(digraph) do
            func(mat, i, j, k);
        od;
    od;
od;
\end{verbatim}

The matrix \textit{mat} is then returned as the result. An example of using \textit{DigraphFloydWarshall} to calculate the shortest (non-zero) distances between the vertices of a digraph is shown below:

\begin{verbatim}
Example
\end{verbatim}

\begin{verbatim}
gap> D := DigraphFromDigraph6String("&EAHQeDB");
<immutable digraph with 6 vertices, 12 edges>
gap> func := function(mat, i, j, k)
>    if mat[i][k] <> -1 and mat[k][j] <> -1 then
>        if (mat[i][j] = -1) or (mat[i][j] > mat[i][k] + mat[k][j]) then
>            mat[i][j] := mat[i][k] + mat[k][j];
>        fi;
>    fi;
\end{verbatim}
Digraphs

> fi;
> end;

function( mat, i, j, k ) ... end

gap> shortest_distances := DigraphFloydWarshall(D, func, -1, 1);

gap> Display(shortest_distances);
[ [ 3, -1, -1, 2, 1, 2 ],
  [ 4, 2, 1, 3, 2, 1 ],
  [ 3, 1, 2, 2, 1, 2 ],
  [ 1, -1, -1, 1, 2, 1 ],
  [ 2, -1, -1, 1, 2, 1 ],
  [ 3, -1, -1, 2, 1, 1 ] ]

5.4.19 IsReachable

▷ IsReachable(diGraph, u, v)

(operation)

Returns: true or false.

This operation returns true if there exists a non-trivial directed walk from vertex u to vertex v in the diGraph diGraph, and false if there does not exist such a directed walk. See Section 1.1.1 for the definition of a non-trivial directed walk.

The method for IsReachable has worst case complexity of $O(m+n)$ where $m$ is the number of edges and $n$ the number of vertices in diGraph.

Example

gap> D := Digraph([1, 1, 1], [2], [3], [4, 3], [2, 3]);
<immutable diGraph with 3 vertices, 4 edges>

gap> IsReachable(D, 1, 3);
true

gap> IsReachable(D, 2, 1);
false

gap> IsReachable(D, 3, 3);
true

gap> IsReachable(D, 1, 1);
false

5.4.20 IsDigraphPath

▷ IsDigraphPath(D, v, a)

▷ IsDigraphPath(D, list)

(operation)

Returns: true or false.

This function returns true if the arguments v and a describe a path in the diGraph D. A directed path (or directed cycle) of non-zero length n-1, $(v_1, e_1, v_2, e_2, ..., e_{n-1}, v_n)$, is represented by a pair of lists $[v, a]$ as follows:

- v is the list $[v_1, v_2, ..., v_n]$.
- a is the list of positive integers $[a_1, a_2, ..., a_{n-1}]$ where for each each $i < n$, $a_i$ is the position of $v_{i+1}$ in OutNeighboursOfVertex(D, $v_i$) corresponding to the edge $e_i$. This can be useful if the position of a vertex in a list of out-neighbours is significant, for example in orbit digraphs.

If the arguments to IsDigraphPath are a diGraph D and list list, then this is equivalent to calling IsDigraphPath(D, list[1], list[2]).
Example

```gap
D := Digraph(IsMutableDigraph, Combinations([1 .. 5]), IsSubset);
<mutable digraph with 32 vertices, 243 edges>
D := DigraphReflexiveTransitiveReduction(D);
<mutable digraph with 32 vertices, 80 edges>
MakeImmutable(D);
<immutable digraph with 32 vertices, 80 edges>
IsDigraphPath(D, [32, 31, 33], [1, 1]);
false
IsDigraphPath(D, [1], []);
true
IsDigraphPath(D, [6, 9, 16, 17], [3, 3, 2]);
true
IsDigraphPath(D, DigraphPath(D, 6, 1));
true
```

5.4.21 VerticesReachableFrom

▷ VerticesReachableFrom(digraph, root)  
Returns: A list.
This operation returns a list of the vertices \( v \), for which there exists a non-trivial directed walk from vertex \( \text{root} \) to vertex \( v \) in the digraph \( \text{digraph} \). See Section 1.1.1 for the definition of a non-trivial directed walk.

The method for VerticesReachableFrom has worst case complexity of \( O(m + n) \) where \( m \) is the number of edges and \( n \) the number of vertices in \( \text{digraph} \).

```gap
D := CompleteDigraph(5);
<immutable complete digraph with 5 vertices>
VerticesReachableFrom(D, 1);
[ 2, 1, 3, 4, 5 ]
VerticesReachableFrom(D, 3);
[ 1, 2, 3, 4, 5 ]
D := EmptyDigraph(5);
<immutable empty digraph with 5 vertices>
VerticesReachableFrom(D, 1);
[ ]
VerticesReachableFrom(D, 3);
[ ]
D := CycleDigraph(4);
<immutable cycle digraph with 4 vertices>
VerticesReachableFrom(D, 1);
[ 2, 3, 4, 1 ]
VerticesReachableFrom(D, 3);
[ 4, 1, 2, 3 ]
D := ChainDigraph(5);
<immutable chain digraph with 5 vertices>
VerticesReachableFrom(D, 1);
[ 2, 3, 4, 5 ]
VerticesReachableFrom(D, 3);
[ 4, 5 ]
VerticesReachableFrom(D, 5);
```
5.4.22 DigraphPath

\[ \text{DigraphPath}(\text{digraph}, u, v) \]

(\text{operation})

\textbf{Returns:} A pair of lists, or fail.

If there exists a non-trivial directed path (or a non-trivial cycle, in the case that \( u = v \)) from vertex \( u \) to vertex \( v \) in the digraph \( \text{digraph} \), then this operation returns such a directed path (or directed cycle). Otherwise, this operation returns fail. See Section ‘Definitions’ for the definition of a directed path and a directed cycle.

A directed path (or directed cycle) of non-zero length \( n-1 \), \((v_1, e_1, v_2, e_2, \ldots, e_{n-1}, v_n)\), is represented by a pair of lists \([v,a]\) as follows:

- \( v \) is the list \([v_1, v_2, \ldots, v_n]\).
- \( a \) is the list of positive integers \([a_1, a_2, \ldots, a_{n-1}]\) where for each each \( i < n \), \( a_i \) is the position of \( v_i+1 \) in \( \text{OutNeighboursOfVertex}(\text{digraph}, v_i) \) corresponding to the edge \( e_i \). This can be useful if the position of a vertex in a list of out-neighbours is significant, for example in orbit digraphs.

The method for DigraphPath has worst case complexity of \( O(m+n) \) where \( m \) is the number of edges and \( n \) the number of vertices in \( \text{digraph} \).

See also IsDigraphPath (5.4.20).

Example

\begin{verbatim}
gap> D := Digraph([[2], [3], [2, 3]]); 
<immutable digraph with 3 vertices, 4 edges>
gap> DigraphPath(D, 1, 3); 
[ [ 1, 3 ], [ 1, 1 ] ]
gap> DigraphPath(D, 2, 1); 
fail
gap> DigraphPath(D, 3, 3); 
[ [ 3, 3 ], [ 2 ] ]
gap> DigraphPath(D, 1, 1); 
fail
\end{verbatim}

5.4.23 DigraphShortestPath

\[ \text{DigraphShortestPath}(\text{digraph}, u, v) \]

(\text{operation})

\textbf{Returns:} A pair of lists, or fail.

Returns a shortest directed path in the digraph \( \text{digraph} \) from vertex \( u \) to vertex \( v \), if such a path exists. If \( u = v \), then the shortest non-trivial cycle is returned, again, if it exists. Otherwise, this operation returns fail. See Section ‘Definitions’ for the definition of a directed path and a directed cycle.

See DigraphPath (5.4.22) for details on the output. The method for DigraphShortestPath has worst case complexity of \( O(m+n) \) where \( m \) is the number of edges and \( n \) the number of vertices in \( \text{digraph} \).

Example

\begin{verbatim}
gap> D := Digraph([[1, 2], [3], [2, 4], [1], [2, 4]]); 
<immutable digraph with 5 vertices, 8 edges>
\end{verbatim}
Digraphs

5.4.24 DigraphRandomWalk

▷ DigraphRandomWalk(digraph, v, t) (operation)

Returns: A pair of lists.

Returns a directed path corresponding to a random walk in the digraph digraph, starting at vertex v and having length no more than t.

A random walk is defined as follows. The path begins at v, and at each step it follows a random edge leaving the current vertex. It continues through the digraph in this way until it has traversed t edges, or until it reaches a vertex with no out-edges (a sink) and therefore cannot continue.

The output has the same form as that of DigraphPath (5.4.22).

Example

gap> D := Digraph([ [1, 2], [3], [2, 4], [1], [2, 4] ]);<immutable digraph with 5 vertices, 8 edges>
gap> DigraphRandomWalk(D, 1, 4);[ [ 1, 2, 3, 2, 3 ], [ 2, 1, 1, 1 ] ]

5.4.25 DigraphAbsorptionProbabilities

▷ DigraphAbsorptionProbabilities(digraph) (attribute)

Returns: A matrix of rational numbers.

A random walk of infinite length through a finite digraph may pass through several different strongly connected components (SCCs). However, with probability 1 it must eventually reach an SCC which it can never leave, because the SCC has no out-edges leading to other SCCs. We may say that such an SCC has absorbed the random walk, and we can calculate the probability of each SCC absorbing a walk starting at a given vertex.

DigraphAbsorptionProbabilities returns an $m \times n$ matrix mat, where $m$ is the number of vertices in digraph and $n$ is the number of strongly connected components. Each entry mat[i][j] is a rational number representing the probability that an unbounded random walk starting at vertex i will be absorbed by strongly connected component j – that is, the probability that it will reach the component and never leave.

Strongly connected components are indexed in the order given by DigraphStronglyConnectedComponents (5.4.11). See also DigraphRandomWalk (5.4.24) and DigraphAbsorptionExpectedSteps (5.4.26).

Example

gap> gr := Digraph([[2, 3, 4], [3], [2], []]);<immutable digraph with 4 vertices, 5 edges>
gap> DigraphStronglyConnectedComponents(gr).comps;[ [ 2, 3 ], [ 4 ], [ 1 ] ]
Digraphs

5.4.26 DigraphAbsorptionExpectedSteps

▷ DigraphAbsorptionExpectedSteps(digraph)  
(attribute)

Returns: A list of rational numbers.

A random walk of unbounded length through a finite digraph may pass through several different strongly connected components (SCCs). However, with probability 1 it must eventually reach an SCC which it can never leave, because the SCC has no out-edges leading to other SCCs. When this happens, we say the walk has been absorbed.

DigraphAbsorptionExpectedSteps returns a list \( L \) with length equal to the number of vertices of \( \text{digraph} \), where \( L[v] \) is the expected number of steps a random walk starting at vertex \( v \) should take before it is absorbed – that is, the expected number of steps to reach an SCC that it can never leave.

See also DigraphRandomWalk (5.4.24) and DigraphAbsorptionProbabilities (5.4.25).

Example

```
gap> gr := Digraph([[2], [3, 4], [], [2]]);
<immutable digraph with 4 vertices, 4 edges>

gap> DigraphAbsorptionExpectedSteps(gr);
[ 4, 3, 0, 4 ]
```

5.4.27 Dominators

▷ Dominators(digraph, root)  
(operation)

Returns: A list of lists.

Dominator takes a digraph and a root \( \text{root} \) and returns the dominators of each vertex with respect to the root. The output is returned as a list of length \( \text{DigraphNrVertices(Digraph)} \), whose \( i \)th entry is a list with the dominators of vertex \( i \) of the \( \text{digraph} \). If there is no path from the root to a specific vertex, the output will contain a hole in the corresponding position. The dominators of a vertex \( u \) are the vertices that are contained in every path from the root to \( u \), not including \( u \) itself.

The method for this operation is an implementation of an algorithm by Thomas Lengauer and Robert Endre Tarjan \([LT79]\). The complexity of this algorithm is \( O(m \log n) \) where \( m \) is the number of edges and \( n \) is the number of nodes in the subdigraph induced by the nodes in \( \text{digraph} \) reachable from \( \text{root} \).

Example

```

gap> D := Digraph([[2], [3, 6], [2, 4], [1], [], [3]]);
<immutable digraph with 6 vertices, 7 edges>

gap> Dominators(D, 1);
[ [ 2, 1 ], [ 3, 2, 1 ], [ 2, 1 ] ]

gap> Dominators(D, 2);
[ [ 4, 2 ], [ 3, 2 ], [ 2 ] ]

gap> Dominators(D, 3);
[ [ 4, 3 ], [ 3 ], [ 2, 3 ] ]

gap> Dominators(D, 4);
[ [ 4 ], [ 1, 4 ], [ 2, 1, 4 ] ]

gap> Dominators(D, 5);
[ ]
```

```
```
5.4.28 DominatorTree

\[ \text{DominatorTree}(\text{digraph}, \text{root}) \]

\textbf{Returns:} A record.

DominatorTree takes a \textit{digraph} and a \textit{root} vertex and returns a record with the following components:

- \textbf{idom} the immediate dominators of the vertices with respect to the root.
- \textbf{preorder} the preorder values of the vertices defined by the depth first search executed on the digraph.

The \textit{immediate dominator} of a vertex \( u \) is the unique dominator of \( u \) that is dominated by all other dominators of \( u \). The algorithm is an implementation of the fast algorithm written by Thomas Lengauer and Robert Endre Tarjan [LT79]. The complexity of this algorithm is \( O(m \log n) \) where \( m \) is the number of edges and \( n \) is the number of nodes in the subdigraph induced by the nodes in \textit{digraph} reachable from \textit{root}.

\[ \text{Example} \]

\begin{verbatim}
gap> D := Digraph([[2, 3], [4, 6], [4, 5], [3, 5], [1, 6], [2, 3]]);
<immutable digraph with 6 vertices, 12 edges>
gap> DominatorTree(D, 1);
rec( idom := [ fail, 1, 1, 1, 1, 1 ],
     preorder := [ 1, 2, 4, 3, 5, 6 ] )
gap> DominatorTree(D, 5);
rec( idom := [ 5, 5, 5, 5, fail, 5 ],
     preorder := [ 5, 1, 2, 4, 3, 6 ] )
gap> D := CompleteDigraph(5);
<immutable complete digraph with 5 vertices>
gap> DominatorTree(D, 1);
rec( idom := [ fail, 1, 1, 1, 1 ], preorder := [ 1, 2, 3, 4, 5 ] )
gap> DominatorTree(D, 2);
rec( idom := [ 2, fail, 2, 2, 2 ], preorder := [ 2, 1, 3, 4, 5 ] )
\end{verbatim}

5.4.29 IteratorOfPaths

\[ \text{IteratorOfPaths}(\text{digraph}, u, v) \]

\textbf{Returns:} An iterator.

If \textit{digraph} is a digraph or a list of adjacencies which defines a digraph - see OutNeighbours (5.2.6) - then this operation returns an iterator of the non-trivial directed paths (or directed cycles, in the case that \( u = v \)) in \textit{digraph} from the vertex \( u \) to the vertex \( v \).

See DigraphPath (5.4.22) for more information about the representation of a directed path or directed cycle which is used, and see (Reference: Iterators) for more information about iterators. See Section ‘Definitions’ for the definition of a directed path and a directed cycle.

\[ \text{Example} \]

\begin{verbatim}
gap> D := Digraph([[1, 4, 4, 2], [3, 5], [2, 3], [1, 2], [4]]);
<immutable multidigraph with 5 vertices, 11 edges>
\end{verbatim}
Digraphs

5.4.30  DigraphAllSimpleCircuits

▷ DigraphAllSimpleCircuits(digraph)  (attribute)

Returns: A list of lists of vertices.

If digraph is a digraph, then DigraphAllSimpleCircuits returns a list of the simple circuits in digraph.

See Section 1.1.1 for the definition of a simple circuit, and related notions. Note that a loop is a simple circuit.

For a digraph without multiple edges, a simple circuit is uniquely determined by its subsequence of vertices. However this is not the case for a multidigraph. The attribute DigraphAllSimpleCircuits ignores multiple edges, and identifies a simple circuit using only its subsequence of vertices. For example, although the simple circuits \((v,e,v)\) and \((v,e',v)\) (for distinct edges \(e\) and \(e'\)) are mathematically distinct, DigraphAllSimpleCircuits considers them to be the same.

With this approach, a directed circuit of length \(n\) can be determined by a list of its first \(n\) vertices. Thus a simple circuit \((v_1,e_1,v_2,e_2,...,e_{n-1},v_n,v_1)\) can be represented as the list \([v_1,...,v_n]\), or any cyclic permutation thereof. For each simple circuit of digraph, DigraphAllSimpleCircuits(digraph) includes precisely one such list to represent the circuit.

Example

```
gap> D := Digraph([[], [3], [2, 4], [5, 4], [4]]);  # immutable digraph with 5 vertices, 6 edges
<immutable digraph>
gap> DigraphAllSimpleCircuits(D);  # simple circuits
[ [ 4 ], [ 4, 5 ], [ 2, 3 ] ]
gap> D := ChainDigraph(10);  # directed chain
<mutable digraph>
gap> DigraphAllSimpleCircuits(D);  # simple circuits
[ ]
gap> D := Digraph([[3], [1], [1]]);  # immutable digraph with 3 vertices, 3 edges
<immutable digraph>
gap> DigraphAllSimpleCircuits(D);  # simple circuits
[ [ 1, 3 ] ]
gap> D := Digraph([[1, 1]]);  # immutable multidigraph with 1 vertex, 2 edges
<immutable multidigraph>
gap> DigraphAllSimpleCircuits(D);  # simple circuits
[ [ 1 ] ]
gap> D := CycleDigraph(IsMutableDigraph, 3);  # mutable digraph with 3 vertices, 3 edges
```
Digraphs

5.4.31 DigraphLongestSimpleCircuit

▷ DigraphLongestSimpleCircuit(digraph)

Returns: A list of vertices, or fail.

If digraph is a digraph, then DigraphLongestSimpleCircuit returns the longest simple circuit in digraph. See Section 1.1.1 for the definition of simple circuit, and the definition of length for a simple circuit.

This attribute computes DigraphAllSimpleCircuits(digraph) to find all the simple circuits of digraph, and returns one of maximal length. A simple circuit is represented as a list of vertices, in the same way as described in DigraphAllSimpleCircuits (5.4.30).

If digraph has no simple circuits, then this attribute returns fail. If digraph has multiple simple circuits of maximal length, then this attribute returns one of them.

Example

gap> D := Digraph([[], [3], [2, 4], [5, 4], [4]]);;
gap> DigraphLongestSimpleCircuit(D);
[ 4, 5 ]
gap> D := ChainDigraph(10);;
gap> DigraphLongestSimpleCircuit(D);
fail
gap> D := Digraph([[3], [1], [1, 4], [1, 1]]);
gap> DigraphLongestSimpleCircuit(D);
[ 1, 3, 4 ]
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 4, 1);
<mutable digraph with 8 vertices, 24 edges>
gap> DigraphLongestSimpleCircuit(D);
[ 1, 2, 3, 4, 8, 7, 6, 5 ]

5.4.32 DigraphLayers

▷ DigraphLayers(digraph, vertex)

Returns: A list.

This operation returns a list list such that list[i] is the list of vertices whose minimum distance from the vertex vertex in digraph is i - 1. Vertex vertex is assumed to be at distance 0 from itself.

Example

gap> D := CompleteDigraph(4);
gap> DigraphLayers(D, 1);
[ [ 1 ], [ 2, 3, 4 ] ]

5.4.33 DigraphDegeneracy

▷ DigraphDegeneracy(digraph)

Returns: A non-negative integer, or fail.

If digraph is a symmetric digraph without multiple edges - see IsSymmetricDigraph (6.2.14) and IsMultiDigraph (6.2.11) - then this attribute returns the degeneracy of digraph.
The degeneracy of a digraph is the least integer \( k \) such that every induced of digraph contains a vertex whose number of neighbours (excluding itself) is at most \( k \). Note that this means that loops are ignored.

If digraph is not symmetric or has multiple edges then this attribute returns fail.

Example

```gap
gap> D := DigraphSymmetricClosure(ChainDigraph(5));;
gap> DigraphDegeneracy(D);
1
gap> D := CompleteDigraph(5);
1
gap> D := Digraph([[1], [2, 4, 5], [3, 4], [2, 3, 4], [2], []]);
<immutable digraph with 6 vertices, 10 edges>
gap> DigraphDegeneracy(D);
1
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 10, 3);
<mutable digraph with 20 vertices, 60 edges>
gap> DigraphDegeneracy(D);
3
```

5.4.34 DigraphDegeneracyOrdering

▶ DigraphDegeneracyOrdering(digraph) (attribute)

Returns: A list of integers, or fail.

If digraph is a digraph for which DigraphDegeneracy(digraph) is a non-negative integer \( k \) - see DigraphDegeneracy (5.4.33) - then this attribute returns a degeneracy ordering of the vertices of the vertices of digraph.

A degeneracy ordering of digraph is a list ordering of the vertices of digraph ordered such that for any position \( i \) of the list, the vertex ordering\( [i] \) has at most \( k \) neighbours in later position of the list.

If DigraphDegeneracy(digraph) returns fail, then this attribute returns fail.

Example

```gap
gap> D := DigraphSymmetricClosure(ChainDigraph(5));;
gap> DigraphDegeneracyOrdering(D);
[ 5, 4, 3, 2, 1 ]
gap> D := CompleteDigraph(5);
[ 5, 4, 3, 2, 1 ]
gap> D := Digraph([[1], [2, 4, 5], [3, 4], [2, 3, 4], [2], []]);
<immutable digraph with 6 vertices, 10 edges>
gap> DigraphDegeneracyOrdering(D);
[ 1, 6, 5, 2, 4, 3 ]
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 3, 1);
<mutable digraph with 6 vertices, 18 edges>
gap> DigraphDegeneracyOrdering(D);
[ 6, 5, 4, 1, 3, 2 ]
```
5.4.35 HamiltonianPath

\[ \text{HamiltonianPath}(\text{digraph}) \]

**Returns:** A list or fail.

Returns a Hamiltonian path if one exists, fail if not.

A Hamiltonian path of a digraph with \( n \) vertices is directed cycle of length \( n \). If \( \text{digraph} \) is a digraph that contains a Hamiltonian path, then this function returns one, described in the form used by \( \text{DigraphAllSimpleCircuits} \) (5.4.30). Note if \( \text{digraph} \) has 0 or 1 vertices, then \( \text{HamiltonianPath} \) returns \([\] \) or \([1]\), respectively.

The method used in this attribute has the same worst case complexity as \( \text{DigraphMonomorphism} \) (7.3.4).

```
gap> D := Digraph([[]]);
<immutable empty digraph with 1 vertex>
gap> HamiltonianPath(D);
[ 1 ]
gap> D := Digraph([[2], [1]]);
<immutable digraph with 2 vertices, 2 edges>
gap> HamiltonianPath(D);
[ 1, 2 ]
gap> D := Digraph([[3], [], [2]]);
<immutable digraph with 3 vertices, 2 edges>
gap> HamiltonianPath(D);
fail
gap> D := Digraph([[2], [3], [1]]);
<immutable digraph with 3 vertices, 3 edges>
gap> HamiltonianPath(D);
[ 1, 2, 3 ]
gap> D := GeneralisedPetersenGraph(IsMutableDigraph, 5, 2);
<mutable digraph with 10 vertices, 30 edges>
gap> HamiltonianPath(D);
fail
```

5.4.36 NrSpanningTrees

\[ \text{NrSpanningTrees}(\text{digraph}) \]

**Returns:** An integer.

Returns the number of spanning trees of the symmetric digraph \( \text{digraph} \). \( \text{NrSpanningTrees} \) will return an error if \( \text{digraph} \) is not a symmetric digraph.

See \( \text{IsSymmetricDigraph} \) (6.2.14) and \( \text{IsUndirectedSpanningTree} \) (4.1.2) for more information.

```
gap> D := CompleteDigraph(5);
<immutable complete digraph with 5 vertices>
gap> NrSpanningTrees(D);
125
gap> D := DigraphSymmetricClosure(CycleDigraph(24));
gap> NrSpanningTrees(D);
24
```
Digraphs

5.4.37 DigraphDijkstra (for a source and target)

▷ DigraphDijkstra(digraph, source, target) (operation)
▷ DigraphDijkstra(digraph, source) (operation)

Returns: Two lists.

If digraph is a digraph and source and target are vertices of digraph, then DigraphDijkstra calculates the length of the shortest path from source to target and returns two lists. Each element of the first list is the distance of the corresponding element from source. If a vertex was not visited in the process of calculating the shortest distance to target or if there is no path connecting that vertex with source, then the corresponding distance is infinity. Each element of the second list gives the previous vertex in the shortest path from source to the corresponding vertex. For source and for any vertices that remained unvisited this will be -1.

If the optional second argument target is not present, then DigraphDijkstra returns the shortest path from source to every vertex that is reachable from source.

Example

gap> mat := [[0, 1, 1], [0, 0, 1], [0, 0, 0]];
[ [ 0, 1, 1 ], [ 0, 0, 1 ], [ 0, 0, 0 ] ]
gap> D := DigraphByAdjacencyMatrix(mat);
<immutable digraph with 3 vertices, 3 edges>
gap> DigraphDijkstra(D, 2, 3);
[ [ infinity, 0, 1 ], [ -1, -1, 2 ] ]
gap> DigraphDijkstra(D, 1, 3);
[ [ 0, 1, 1 ], [ -1, 1, 1 ] ]
gap> DigraphDijkstra(D, 1, 2);
[ [ 0, 1, 1 ], [ -1, 1, 1 ] ]

5.5 Cayley graphs of groups

5.5.1 GroupOfCayleyDigraph

▷ GroupOfCayleyDigraph(digraph) (attribute)
▷ SemigroupOfCayleyDigraph(digraph) (attribute)

Returns: A group or semigroup.

If digraph is an immutable Cayley graph of a group G and digraph belongs to the category IsCayleyDigraph (3.1.4), then GroupOfCayleyDigraph returns G.

If digraph is a Cayley graph of a semigroup S and digraph belongs to the category IsCayleyDigraph (3.1.4), then SemigroupOfCayleyDigraph returns S.

See also GeneratorsOfCayleyDigraph (5.5.2).

Example

gap> G := DihedralGroup(IsPermGroup, 8);
Group([ (1,2,3,4), (2,4) ])
gap> digraph := CayleyDigraph(G);
<immutable digraph with 8 vertices, 16 edges>
5.5.2 GeneratorsOfCayleyDigraph

GeneratorsOfCayleyDigraph(digraph) (attribute)

Returns: A list of generators.

If digraph is an immutable Cayley graph of a group or semigroup with respect to a set of generators gens and digraph belongs to the category IsCayleyDigraph (3.1.4), then GeneratorsOfCayleyDigraph return the list of generators gens over which digraph is defined.

See also GroupOfCayleyDigraph (5.5.1) or SemigroupOfCayleyDigraph (5.5.1).

Example:

```gap
    gap> G := DihedralGroup(IsPermGroup, 8);
    Group([ (1,2,3,4), (2,4) ])
    gap> digraph := CayleyDigraph(G);
    <immutable digraph with 8 vertices, 16 edges>
    gap> GeneratorsOfCayleyDigraph(digraph) = GeneratorsOfGroup(G);
    true
    gap> digraph := CayleyDigraph(G, [()]);
    <immutable digraph with 8 vertices, 8 edges>
    gap> GeneratorsOfCayleyDigraph(digraph) = [()];
    true
```

5.6 Associated semigroups

5.6.1 AsSemigroup (for a filter and a digraph)

AsSemigroup(filt, digraph) (operation)

AsMonoid(filt, digraph) (operation)

Returns: A semilattice of partial perms.

The operation AsSemigroup requires that filt be equal to IsPartialPermSemigroup (Reference: IsPartialPermSemigroup). If digraph is a IsJoinSemilatticeDigraph (6.4.3) or IsLatticeDigraph (6.4.3) then AsSemigroup returns a semigroup of partial perms which is isomorphic to the semigroup whose elements are the vertices of digraph with the binary operation PartialOrderDigraphJoinOfVertices (5.3.1). If digraph satisfies IsMeetSemilatticeDigraph (6.4.3) but not IsJoinSemilatticeDigraph (6.4.3) then AsSemigroup returns a semigroup of partial perms which is isomorphic to the semigroup whose elements are the vertices of digraph with the binary operation PartialOrderDigraphMeetOfVertices (5.3.1).

The operation AsMonoid behaves similarly to AsSemigroup except that filt may also be equal to IsPartialPermMonoid (Reference: IsPartialPermMonoid), digraph must satisfy IsLatticeDigraph (6.4.3), and the output satisfies IsMonoid (Reference: IsMonoid).

The output of both of these operations is guaranteed to be of minimal degree (see DegreeOfPartialPermSemigroup (Reference: DegreeOfPartialPermSemigroup)). Furthermore the GeneratorsOfSemigroup (Reference: GeneratorsOfSemigroup) of the output is guaranteed to be the unique generating set of minimal size.
Example

```gap
gap> di := Digraph([[1], [1, 2], [1, 3], [1, 4], [1, 2, 3, 5]]);
<immutable digraph with 5 vertices, 11 edges>
gap> S := AsSemigroup(IsPartialPermSemigroup, di);
<partial perm semigroup of rank 3 with 4 generators>
gap> ForAll(Elements(S), IsIdempotent);
true
gap> IsInverseSemigroup(S);
true
gap> Size(S);
5
gap> di := Digraph([[1], [1, 2], [1, 2, 3]]);
<immutable digraph with 3 vertices, 6 edges>
gap> M := AsMonoid(IsPartialPermMonoid, di);
<partial perm monoid of rank 2 with 3 generators>
gap> Size(M);
3
```

### 5.6.2 AsSemigroup (for a filter, semilattice digraph, and two lists)

The operation `AsSemigroup` requires that `filt` be equal to `IsPartialPermSemigroup` (Reference: `IsPartialPermSemigroup`). If `Y` is a `IsJoinSemilatticeDigraph` (6.4.3) or `IsMeetSemilatticeDigraph` (6.4.3), `gps` is a list of groups corresponding to each vertex, and `homs` is a list containing for each edge \((i, j)\) in the transitive reduction of `digraph` a triple \([i, j, \text{hom}]\) where `hom` is a group homomorphism from `gps[i]` to `gps[j]`, and the diagram of homomorphisms commutes, then `AsSemigroup` returns a semigroup of partial perms which is isomorphic to the strong semilattice of groups \(S[Y; gps; homs]\).

Example

```gap
gap> G1 := AlternatingGroup(4);;
gap> G2 := SymmetricGroup(2);;
gap> G3 := SymmetricGroup(3);;
gap> gr := Digraph([[1, 3], [2, 3], [3]]);
gap> sgn := function(x)
> if SignPerm(x) = 1 then
> return ();
> fi;
> return (1, 2);
> end;
> hom13 := GroupHomomorphismByFunction(G1, G3, sgn);;
gap> hom23 := GroupHomomorphismByFunction(G2, G3, sgn);;
gap> T := AsSemigroup(IsPartialPermSemigroup,
> gr,
> [G1, G2, G3], [[1, 3, hom13], [2, 3, hom23]]);
gap> Size(T);
20
gap> D := GreensDClasses(T);;
gap> List(D, Size);
[ 6, 12, 2 ]
```
5.7 Planarity

5.7.1 KuratowskiPlanarSubdigraph

\[ \text{KuratowskiPlanarSubdigraph}(\text{digraph}) \]

**Returns:** A list or fail.

KuratowskiPlanarSubdigraph returns the immutable list of lists of out-neighbours of a (not necessarily induced) subdigraph of the digraph \textit{digraph} that witnesses the fact that \textit{digraph} is not planar, or fail if \textit{digraph} is planar. In other words, KuratowskiPlanarSubdigraph returns the out-neighbours of a subdigraph of \textit{digraph} that is homeomorphic to the complete graph with 5 vertices, or to the complete bipartite graph with vertex sets of sizes 3 and 3.

The directions and multiplicities of any edges in \textit{digraph} are ignored when considering whether or not \textit{digraph} is planar.

See also \textit{IsPlanarDigraph} (6.7.1) and \textit{SubdigraphHomeomorphicToK33} (5.7.5).

This method uses the reference implementation in \textit{edge-addition-planarity-suite} by John Boyer of the algorithms described in [BM06].

**Example**

```gap
gap> D := Digraph([[3, 5, 10], [8, 9, 10], [1, 4], [3, 6],
>     [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<immutable digraph with 11 vertices, 25 edges>
gap> KuratowskiPlanarSubdigraph(D);
fail

gap> D := Digraph([[2, 4, 7, 9, 10], [1, 3, 4, 6, 9, 10], [6, 10],
>     [2, 5, 8, 9], [1, 2, 3, 4, 6, 7, 9, 10], [3, 4, 5, 7, 9, 10],
>     [3, 4, 5, 6, 9, 10], [3, 4, 5, 7, 9], [2, 3, 5, 6, 7, 8], [3, 5]]);
<immutable digraph with 10 vertices, 50 edges>
gap> IsPlanarDigraph(D);
false

gap> KuratowskiPlanarSubdigraph(D);
[ [ 2, 9, 7 ], [ 3 ], [ 6 ], [ 5, 9 ], [ 6 ], [ ], [ 4 ],
  [ 7, 9, 3 ], [ ], [ ] ]

gap> D := Digraph(IsMutableDigraph, [[3, 5, 10], [8, 9, 10], [1, 4],
>     [3, 6], [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<mutable digraph with 11 vertices, 25 edges>
gap> KuratowskiPlanarSubdigraph(D);
fail

gap> D := Digraph(IsMutableDigraph, [[2, 4, 7, 9, 10],
>     [1, 3, 4, 6, 9, 10], [6, 10], [2, 5, 8, 9],
>     [1, 2, 3, 4, 6, 7, 9, 10], [3, 4, 5, 7, 9, 10],
>     [3, 4, 5, 6, 9, 10], [3, 4, 5, 7, 9], [2, 3, 5, 6, 7, 8], [3, 5]]);
<mutable digraph with 10 vertices, 50 edges>
gap> IsPlanarDigraph(D);
false

gap> KuratowskiPlanarSubdigraph(D);
[ [ 2, 9, 7 ], [ 3 ], [ 6 ], [ 5, 9 ], [ 6 ], [ ], [ 4 ],
  [ 7, 9, 3 ], [ ], [ ] ]
```

5.7.2 KuratowskiOuterPlanarSubdigraph

\[ \text{KuratowskiOuterPlanarSubdigraph}(\text{digraph}) \]

**Returns:** A list or fail.

KuratowskiOuterPlanarSubdigraph returns the outerplanar subdigraph of \textit{digraph} that witnesses the fact that \textit{digraph} is not planar, or fail if \textit{digraph} is planar. In other words, KuratowskiOuterPlanarSubdigraph returns the outerplanar subdigraph of \textit{digraph} that is homeomorphic to the complete graph with 5 vertices, or to the complete bipartite graph with vertex sets of sizes 3 and 3.

The directions and multiplicities of any edges in \textit{digraph} are ignored when considering whether or not \textit{digraph} is planar.

See also \textit{IsPlanarDigraph} (6.7.1) and \textit{SubdigraphHomeomorphicToK33} (5.7.5).

This method uses the reference implementation in \textit{edge-addition-planarity-suite} by John Boyer of the algorithms described in [BM06].

**Example**

```gap
gap> D := Digraph([[3, 5, 10], [8, 9, 10], [1, 4], [3, 6],
>     [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<immutable digraph with 11 vertices, 25 edges>
gap> KuratowskiOuterPlanarSubdigraph(D);
fail

gap> D := Digraph([[2, 4, 7, 9, 10], [1, 3, 4, 6, 9, 10], [6, 10],
>     [2, 5, 8, 9], [1, 2, 3, 4, 6, 7, 9, 10], [3, 4, 5, 7, 9, 10],
>     [3, 4, 5, 6, 9, 10], [3, 4, 5, 7, 9], [2, 3, 5, 6, 7, 8], [3, 5]]);
<immutable digraph with 10 vertices, 50 edges>
gap> IsPlanarDigraph(D);
false

gap> KuratowskiOuterPlanarSubdigraph(D);
[ [ 2, 9, 7 ], [ 3 ], [ 6 ], [ 5, 9 ], [ 6 ], [ ], [ 4 ],
  [ 7, 9, 3 ], [ ], [ ] ]
```
KuratowskiOuterPlanarSubdigraph returns the immutable list of immutable lists of out-neighbours of a (not necessarily induced) subdigraph of the digraph \textit{digraph} that witnesses the fact that \textit{digraph} is not outer planar, or fail if \textit{digraph} is outer planar. In other words, KuratowskiOuterPlanarSubdigraph returns the out-neighbours of a subdigraph of \textit{digraph} that is homeomorphic to the complete graph with 4 vertices, or to the complete bipartite graph with vertex sets of sizes 2 and 3.

The directions and multiplicities of any edges in \textit{digraph} are ignored when considering whether or not \textit{digraph} is outer planar.

See also IsOuterPlanarDigraph (6.7.2), SubdigraphHomeomorphicToK4 (5.7.5), and SubdigraphHomeomorphicToK23 (5.7.5).

This method uses the reference implementation in edge-addition-planarity-suite by John Boyer of the algorithms described in [BM06].

Example

\begin{verbatim}
gap> D := Digraph([[3, 5, 10], [8, 9, 10], [1, 4], [3, 6],
> [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<immutable digraph with 11 vertices, 25 edges>
gap> KuratowskiOuterPlanarSubdigraph(D);
[ [ 3, 5, 10 ], [ 9, 8, 10 ], [ 4 ], [ 6 ], [ 11 ], [ 7 ], [ 8 ],
 [ ], [ 11 ], [ ], [ ] ]
gap> D := Digraph([[2, 4, 7, 9, 10], [1, 3, 4, 6, 9, 10], [6, 10],
> [2, 5, 8, 9], [1, 2, 3, 4, 6, 7, 9, 10], [3, 4, 5, 7, 9, 10],
> [3, 4, 5, 6, 9, 10], [3, 4, 5, 7, 9], [2, 3, 5, 6, 7, 8], [3, 5]]);
<immutable digraph with 10 vertices, 50 edges>
gap> IsOuterPlanarDigraph(D);
false
gap> KuratowskiOuterPlanarSubdigraph(D);
[ [ ], [ ], [ ], [ 8, 9 ], [ ], [ ], [ 9, 4 ], [ 7, 9 ], [ ],
 [ ], [ 11 ], [ ], [ ] ]
gap> D := Digraph(IsMutableDigraph, [[3, 5, 10], [8, 9, 10], [1, 4],
> [3, 6], [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<mutable digraph with 11 vertices, 25 edges>
gap> KuratowskiOuterPlanarSubdigraph(D);
[ [ 3, 5, 10 ], [ 9, 8, 10 ], [ 4 ], [ 6 ], [ 11 ], [ 7 ], [ 8 ],
 [ ], [ 11 ], [ ], [ ] ]
gap> D := Digraph(IsMutableDigraph, [[2, 4, 7, 9, 10],
> [1, 3, 4, 6, 9, 10], [6, 10], [2, 5, 8, 9],
> [1, 2, 3, 4, 6, 7, 9, 10], [3, 4, 5, 7, 9, 10],
> [3, 4, 5, 6, 9, 10], [3, 4, 5, 7, 9], [2, 3, 5, 6, 7, 8], [3, 5]]);
<mutable digraph with 10 vertices, 50 edges>
gap> IsOuterPlanarDigraph(D);
false
gap> KuratowskiOuterPlanarSubdigraph(D);
[ [ ], [ ], [ ], [ 8, 9 ], [ ], [ ], [ 9, 4 ], [ 7, 9 ], [ ],
 [ ], [ 11 ], [ ], [ ] ]
\end{verbatim}

5.7.3 PlanarEmbedding

\begin{verbatim}
PlanarEmbedding(digraph) (attribute)
\end{verbatim}

\textbf{Returns:} A list or fail.
If \textit{digraph} is a planar digraph, then \texttt{PlanarEmbedding} returns the immutable list of lists of out-neighbours of a subdigraph of \textit{digraph} such that each vertex’s neighbours are given in clockwise order. If \textit{digraph} is not planar, then \texttt{fail} is returned.

The directions and multiplicities of any edges in \textit{digraph} are ignored by \texttt{PlanarEmbedding}.

See also \texttt{IsPlanarDigraph (6.7.1)}.

This method uses the reference implementation in edge-addition-planarity-suite by John Boyer of the algorithms described in [BM06].

\begin{example}
\begin{verbatim}
gap> D := Digraph([[3, 5, 10], [8, 9, 10], [1, 4], [3, 6],
  > [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<immutable digraph with 11 vertices, 25 edges>
\end{verbatim}
\end{example}

\begin{example}
\begin{verbatim}
gap> PlanarEmbedding(D);
[ [ 3, 10, 5 ], [ 10, 8, 9 ], [ 4 ], [ 6 ], [ 11, 7 ], [ 7 ], [ 8 ],
  [ ], [ 11 ], [ ], [ ] ]
\end{verbatim}
\end{example}

\begin{example}
\begin{verbatim}
gap> D := Digraph(IsMutableDigraph, [[3, 5, 10], [8, 9, 10], [1, 4],
  > [3, 6], [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<mutable digraph with 11 vertices, 25 edges>
\end{verbatim}
\end{example}

5.7.4 \texttt{OuterPlanarEmbedding}

\begin{verbatim}
}\texttt{OuterPlanarEmbedding(digraph)} (attribute)
\end{verbatim}

\textbf{Returns:} A list or \texttt{fail}.

If \textit{digraph} is an outer planar digraph, then \texttt{OuterPlanarEmbedding} returns the immutable list of lists of out-neighbours of a subdigraph of \textit{digraph} such that each vertex’s neighbours are given in clockwise order. If \textit{digraph} is not outer planar, then \texttt{fail} is returned.

The directions and multiplicities of any edges in \textit{digraph} are ignored by \texttt{OuterPlanarEmbedding}.

See also \texttt{IsOuterPlanarDigraph (6.7.2)}.

This method uses the reference implementation in edge-addition-planarity-suite by John Boyer of the algorithms described in [BM06].

\begin{example}
\begin{verbatim}
gap> D := Digraph([[3, 5, 10], [8, 9, 10], [1, 4], [3, 6],
  > [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<immutable digraph with 11 vertices, 25 edges>
\end{verbatim}
\end{example}
5.7.5 SubdigraphHomeomorphicToK23

These attributes return the immutable list of lists of out-neighbours of a subdigraph of the digraph `digraph` which is homeomorphic to one of the following: the complete bipartite graph with vertex sets of sizes 2 and 3; the complete bipartite graph with vertex sets of sizes 3 and 3; or the complete graph with 4 vertices. If `digraph` has no such subdigraphs, then `fail` is returned.

See also `IsPlanarDigraph (6.7.1)` and `IsOuterPlanarDigraph (6.7.2)` for more details.

This method uses the reference implementation in edge-addition-planarity-suite by John Boyer of the algorithms described in [BM06].
Digraphs

gap> SubdigraphHomeomorphicToK4(D);
fail

gap> SubdigraphHomeomorphicToK23(D);
[ [ 3, 10, 5 ], [ 10, 8, 9 ], [ 4 ], [ 6 ], [ 11 ], [ 7 ], [ 8 ],
  [ ], [ 11 ], [ ], [ ] ]
gap> SubdigraphHomeomorphicToK33(D);
fail

gap> SubdigraphHomeomorphicToK23(NullDigraph(0));
fail

gap> SubdigraphHomeomorphicToK33(CompleteDigraph(5));
fail

gap> SubdigraphHomeomorphicToK33(CompleteBipartiteDigraph(3, 3));
[ [ 4, 6, 5 ], [ 4, 5, 6 ], [ 6, 5, 4 ], [ ], [ ], [ ] ]
gap> SubdigraphHomeomorphicToK4(CompleteDigraph(3));
fail

gap> D := Digraph(IsMutableDigraph, [[3, 5, 10], [8, 9, 10], [1, 4],
> [3, 6], [1, 7, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<mutable digraph with 11 vertices, 25 edges>
gap> SubdigraphHomeomorphicToK4(D);
[ [ 3, 5, 10 ], [ 9, 8, 10 ], [ 4 ], [ 6 ], [ 7, 11 ], [ 7 ], [ 8 ],
  [ ], [ 11 ], [ ], [ ] ]
gap> SubdigraphHomeomorphicToK23(D);
[ [ 3, 5, 10 ], [ 9, 8, 10 ], [ 4 ], [ 6 ], [ 11 ], [ 7 ], [ 8 ],
  [ ], [ 11 ], [ ], [ ] ]
gap> D := Digraph(IsMutableDigraph, [[3, 5, 10], [8, 9, 10], [1, 4],
> [3, 6], [1, 11], [4, 7], [6, 8], [2, 7], [2, 11], [1, 2], [5, 9]]);
<mutable digraph with 11 vertices, 24 edges>
gap> SubdigraphHomeomorphicToK4(D);
fail

gap> SubdigraphHomeomorphicToK23(D);
[ [ 3, 10, 5 ], [ 10, 8, 9 ], [ 4 ], [ 6 ], [ 11 ], [ 7 ], [ 8 ],
  [ ], [ 11 ], [ ], [ ] ]
gap> SubdigraphHomeomorphicToK33(D);
fail

gap> SubdigraphHomeomorphicToK23(NullDigraph(0));
fail

gap> SubdigraphHomeomorphicToK33(CompleteDigraph(5));
fail

gap> SubdigraphHomeomorphicToK33(CompleteBipartiteDigraph(3, 3));
[ [ 4, 6, 5 ], [ 4, 5, 6 ], [ 6, 5, 4 ], [ ], [ ], [ ] ]
gap> SubdigraphHomeomorphicToK4(CompleteDigraph(3));
fail
Chapter 6

Properties of digraphs

6.1  Vertex properties

6.1.1  DigraphHasAVertex

\[\text{DigraphHasAVertex}(\text{digraph})\]

**Returns:** true or false.

Returns true if the digraph \textit{digraph} has at least one vertex, and false if it does not.

See also DigraphHasNoVertices (6.1.2).

If the argument \textit{digraph} is mutable, then the return value of this property is recomputed every time it is called.

\begin{verbatim}
gap> D := Digraph([]); <immutable empty digraph with 0 vertices> gap> DigraphHasAVertex(D); false gap> D := Digraph([[1]]); <immutable empty digraph with 1 vertex> gap> DigraphHasAVertex(D); true gap> D := Digraph([], [1]); <immutable digraph with 2 vertices, 1 edge> gap> DigraphHasAVertex(D); true
\end{verbatim}

6.1.2  DigraphHasNoVertices

\[\text{DigraphHasNoVertices}(\text{digraph})\]

**Returns:** true or false.

Returns true if the digraph \textit{digraph} is the unique digraph with zero vertices, and false otherwise.

See also DigraphHasAVertex (6.1.1).

If the argument \textit{digraph} is mutable, then the return value of this property is recomputed every time it is called.

\begin{verbatim}
gap> D := Digraph([]); <immutable empty digraph with 0 vertices>
\end{verbatim}
6.2 Edge properties

6.2.1 DigraphHasLoops

\[\text{DigraphHasLoops}(\text{digraph})\]

**Returns:** true or false.

Returns true if the digraph \text{digraph} has loops, and false if it does not. A loop is an edge with equal source and range.

If the argument \text{digraph} is mutable, then the return value of this property is recomputed every time it is called.

```
gap> D := Digraph([[1, 2], [2]]);  
<immutable digraph with 2 vertices, 3 edges>  
gap> DigraphEdges(D);  
[ [ 1, 1 ], [ 1, 2 ], [ 2, 2 ] ]  
gap> DigraphHasLoops(D);  
true  
gap> D := Digraph([[2, 3], [1], [2]]);  
<immutable digraph with 3 vertices, 4 edges>  
gap> DigraphEdges(D);  
[ [ 1, 2 ], [ 1, 3 ], [ 2, 1 ], [ 3, 2 ] ]  
gap> DigraphHasLoops(D);  
false  
gap> D := CompleteDigraph(IsMutableDigraph, 4);  
<mutable digraph with 4 vertices, 12 edges>  
gap> DigraphHasLoops(D);  
false
```

6.2.2 IsAntiSymmetricDigraph

\[\text{IsAntiSymmetricDigraph}(\text{digraph})\]

\[\text{IsAntiSymmetricDigraph}(\text{digraph})\]

**Returns:** true or false.

This property is true if the digraph \text{digraph} is antisymmetric, and false if it is not.

A digraph is antisymmetric if whenever there is an edge with source \(u\) and range \(v\), and an edge with source \(v\) and range \(u\), then the vertices \(u\) and \(v\) are equal.

If the argument \text{digraph} is mutable, then the return value of this property is recomputed every time it is called.
Digraphs

Example

gap> gr1 := Digraph([[2], [1, 3], [2, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsAntisymmetricDigraph(gr1);
false
gap> DigraphEdges(gr1){[1, 2]};
[ [ 1, 2 ], [ 2, 1 ] ]

Example

gap> gr2 := Digraph([[1, 2], [3, 3], [1]]);
<immutable multidigraph with 3 vertices, 5 edges>
gap> IsAntisymmetricDigraph(gr2);
true
gap> DigraphEdges(gr2);
[ [ 1, 1 ], [ 1, 2 ], [ 2, 3 ], [ 2, 3 ], [ 3, 1 ] ]

6.2.3 IsBipartiteDigraph

▷ IsBipartiteDigraph(digraph) (property)

Returns: true or false.

This property is true if the digraph digraph is bipartite, and false if it is not. A digraph is bipartite if and only if the vertices of digraph can be partitioned into two non-empty sets such that the source and range of any edge of digraph lie in distinct sets. Equivalently, a digraph is bipartite if and only if it is 2-colorable; see DigraphGreedyColouring (7.3.15).

See also DigraphBicomponents (5.4.13).

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

Example

gap> D := ChainDigraph(4);
<immutable chain digraph with 4 vertices>
gap> IsBipartiteDigraph(D);
true
gap> D := CycleDigraph(3);
<immutable cycle digraph with 3 vertices>
gap> IsBipartiteDigraph(D);
true
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 5, 4);
<mutable digraph with 9 vertices, 40 edges>
gap> IsBipartiteDigraph(D);
true

6.2.4 IsCompleteBipartiteDigraph

▷ IsCompleteBipartiteDigraph(digraph) (property)

Returns: true or false.

Returns true if the digraph digraph is a complete bipartite digraph, and false if it is not.

A digraph is a complete bipartite digraph if it is bipartite, see IsBipartiteDigraph (6.2.3), and there exists a unique edge with source i and range j if and only if i and j lie in different bicomponents of digraph, see DigraphBicomponents (5.4.13).

Equivalently, a bipartite digraph with bicomponents of size m and n is complete precisely when it has $2mn$ edges, none of which are multiple edges.

See also CompleteBipartiteDigraph (3.5.12).

If the argument \texttt{digraph} is mutable, then the return value of this property is recomputed every
time it is called.

```gap
gap> D := CycleDigraph(2);
<immutable cycle digraph with 2 vertices>
gap> IsCompleteBipartiteDigraph(D);  # true
<immutable cycle digraph with 2 vertices>
gap> D := CycleDigraph(4);
<immutable cycle digraph with 4 vertices>
gap> IsBipartiteDigraph(D);  # true
<immutable cycle digraph with 4 vertices>
gap> IsCompleteBipartiteDigraph(D);  # false
<immutable cycle digraph with 4 vertices>
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 5, 4);
<mutable digraph with 9 vertices, 40 edges>
gap> IsCompleteBipartiteDigraph(D);  # true
```

### 6.2.5 IsCompleteDigraph

\textbf{Returns:} true or false.

Returns true if the digraph \texttt{digraph} is complete, and false if it is not.

A digraph is complete if it has no loops, and for all distinct vertices \(i\) and \(j\), there is exactly one
edge with source \(i\) and range \(j\). Equivalently, a digraph with \(n\) vertices is complete precisely when it
has \(n(n - 1)\) edges, no loops, and no multiple edges.

If the argument \texttt{digraph} is mutable, then the return value of this property is recomputed every
time it is called.

```gap
gap> D := Digraph([[2, 3], [1, 3], [1, 2]]);
<immutable digraph with 3 vertices, 6 edges>
gap> IsCompleteDigraph(D);  # true
<immutable digraph with 3 vertices, 6 edges>
gap> D := Digraph([[2, 2], [1]]);
<immutable multidigraph with 2 vertices, 3 edges>
gap> IsCompleteDigraph(D);  # false
<immutable multidigraph with 2 vertices, 3 edges>
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 5, 4);
<mutable digraph with 9 vertices, 40 edges>
gap> IsCompleteDigraph(D);  # false
```

### 6.2.6 IsCompleteMultipartiteDigraph

\textbf{Returns:} true or false.

This property returns true if \texttt{digraph} is a complete multipartite digraph, and false if not.
A digraph is a complete multipartite digraph if and only if its vertices can be partitioned into at least two maximal independent sets, where every possible edge between these independent sets occurs in the digraph exactly once.

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

Example

```gap
gap> D := CompleteMultipartiteDigraph([2, 4, 6]);
<immutable complete multipartite digraph with 12 vertices, 88 edges>
gap> IsCompleteMultipartiteDigraph(D);
true
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 5, 4);
<mutable digraph with 9 vertices, 40 edges>
gap> IsCompleteMultipartiteDigraph(D);
true
```

### 6.2.7 IsEmptyDigraph

- **IsEmptyDigraph(digraph)** (property)
- **IsNullDigraph(digraph)** (property)

**Returns:** true or false.

Returns `true` if the digraph `digraph` is empty, and `false` if it is not. A digraph is `empty` if it has no edges. `IsNullDigraph` is a synonym for `IsEmptyDigraph`. See also `IsNonemptyDigraph` (6.2.12).

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

Example

```gap
gap> D := Digraph([[], []]);
<immutable empty digraph with 2 vertices>
gap> IsEmptyDigraph(D);
true
gap> IsNullDigraph(D);
true
gap> D := Digraph([[], [1]]);
<immutable digraph with 2 vertices, 1 edge>
gap> IsEmptyDigraph(D);
false
gap> IsNullDigraph(D);
false
```

### 6.2.8 IsEquivalenceDigraph

- **IsEquivalenceDigraph(digraph)** (property)

**Returns:** true or false.

A digraph is an equivalence digraph if and only if the digraph satisfies all of `IsReflexiveDigraph` (6.2.13), `IsSymmetricDigraph` (6.2.14) and `IsTransitiveDigraph` (6.2.16). A partial order `digraph` corresponds to an equivalence relation.

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.
6.2.9  IsFunctionalDigraph

▷ IsFunctionalDigraph(digraph)  

Returns: true or false.

This property is true if the digraph digraph is functional.  
A digraph is functional if every vertex is the source of a unique edge.  
If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

Example

\begin{verbatim}
gap> gr1 := Digraph([[3], [2], [2], [1], [6], [5]]);  
<immutable digraph with 6 vertices, 6 edges>  
gap> IsFunctionalDigraph(gr1);  
true  
gap> gr2 := Digraph([[1, 2], [1]]);  
<immutable digraph with 2 vertices, 3 edges>  
gap> IsFunctionalDigraph(gr2);  
false  
gap> gr3 := Digraph(3, [1, 2, 3], [2, 3, 1]);  
<immutable digraph with 3 vertices, 3 edges>  
gap> IsFunctionalDigraph(gr3);  
true
\end{verbatim}

6.2.10  IsPermutationDigraph

▷ IsPermutationDigraph(digraph)  

Returns: true or false.

This property is true if the digraph digraph is functional and each node has only one in-neighbour.  
If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

Example

\begin{verbatim}
gap> gr1 := Digraph([[3], [2], [2], [1], [6], [5]]);  
<immutable digraph with 6 vertices, 6 edges>  
gap> IsPermutationDigraph(gr1);  
false  
gap> gr2 := Digraph([[1, 2], [1]]);  
<immutable digraph with 2 vertices, 3 edges>  
gap> IsPermutationDigraph(gr2);  
false  
gap> gr3 := Digraph(3, [1, 2, 3], [2, 3, 1]);  
<immutable digraph with 3 vertices, 3 edges>  
gap> IsPermutationDigraph(gr3);  
true
\end{verbatim}
6.2.11 IsMultiDigraph

▷ IsMultiDigraph(digraph)  

Returns: true or false.

A multidigraph is one that has at least two edges with equal source and range.

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

Example

```gap
D := Digraph(["a", "b", "c"], ["a", "b", "b"], ["b", "c", "a"]);
<immutable digraph with 3 vertices, 3 edges>
IsMultiDigraph(D);
false
D := DigraphFromDigraph6String("&Bug");
<immutable digraph with 3 vertices, 6 edges>
IsDuplicateFree(DigraphEdges(D));
true
IsMultiDigraph(D);
false
D := Digraph([[1, 2, 3, 2], [2, 1], [3]]);
<immutable multidigraph with 3 vertices, 7 edges>
IsDuplicateFree(DigraphEdges(D));
false
IsMultiDigraph(D);
true
D := DigraphMutableCopy(D);
<mutable multidigraph with 3 vertices, 7 edges>
IsMultiDigraph(D);
true
```

6.2.12 IsNonemptyDigraph

▷ IsNonemptyDigraph(digraph)  

Returns: true or false.

Returns true if the digraph digraph is nonempty, and false if it is not. A digraph is nonempty if it has at least one edge.

See also IsEmptyDigraph (6.2.7).

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

Example

```gap
D := Digraph([[], []]);
<immutable empty digraph with 2 vertices>
IsNonemptyDigraph(D);
false
D := Digraph([[], [1]]);
<immutable digraph with 2 vertices, 1 edge>
IsNonemptyDigraph(D);
true
```
6.2.13 IsReflexiveDigraph

▷ IsReflexiveDigraph(digraph)  
(property)

Returns: true or false.

This property is true if the digraph digraph is reflexive, and false if it is not. A digraph is reflexive if it has a loop at every vertex.

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

Example

\[ \text{gap> } D := \text{Digraph}([[1, 2], [2]]); \\
\text{<immutable digraph with 2 vertices, 3 edges>} \\
\text{gap> } \text{IsReflexiveDigraph}(D); \\
\text{true} \\
\text{gap> } D := \text{Digraph}([[3, 1], [4, 2], [3], [2, 1]]); \\
\text{<immutable digraph with 4 vertices, 7 edges>} \\
\text{gap> } \text{IsReflexiveDigraph}(D); \\
\text{false} \]

6.2.14 IsSymmetricDigraph

▷ IsSymmetricDigraph(digraph)  
(property)

Returns: true or false.

This property is true if the digraph digraph is symmetric, and false if it is not.

A symmetric digraph is one where for each non-loop edge, having source u and range v, there is a corresponding edge with source v and range u. If there are n edges with source u and range v, then there must be precisely n edges with source v and range u. In other words, a symmetric digraph has a symmetric adjacency matrix AdjacencyMatrix (5.2.1).

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

Example

\[ \text{gap> } \text{gr1 := Digraph}([[2], [1, 3], [2, 3]]); \\
\text{<immutable digraph with 3 vertices, 5 edges>} \\
\text{gap> } \text{IsSymmetricDigraph}(\text{gr1}); \\
\text{true} \\
\text{gap> } \text{adj1 := AdjacencyMatrix} (\text{gr1}); \\
\text{true} \\
\text{gap> } \text{Display}(\text{adj1}); \\
\text{[ [ 0, 1, 0 ],} \\
\text{ [ 1, 0, 1 ],} \\
\text{ [ 0, 1, 1 ] ]} \\
\text{gap> } \text{adj1 = TransposedMat} (\text{adj1}); \\
\text{true} \\
\text{gap> } \text{gr1 = DigraphReverse} (\text{gr1}); \\
\text{true} \\
\text{gap> } \text{gr2 := Digraph}([[2, 3], [1, 3], [2, 3]]); \\
\text{<immutable digraph with 3 vertices, 6 edges>} \\
\text{gap> } \text{IsSymmetricDigraph}(\text{gr2}); \\
\text{false} \\
\text{gap> } \text{adj2 := AdjacencyMatrix} (\text{gr2}); \\
\text{true} \\
\text{gap> } \text{Display}(\text{adj2}); \\
\text{[ [ 0, 1, 1 ],} \\
\text{ [ 1, 0, 1 ] ]} \]
6.2.15  IsTournament

▷ IsTournament(digraph)  

Returns: true or false.

This property is true if the digraph digraph is a tournament, and false if it is not. A tournament is an orientation of a complete (undirected) graph. Specifically, a tournament is a digraph which has a unique directed edge (of some orientation) between any pair of distinct vertices, and no loops.

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

Example

```
gap> D := Digraph([[2, 3, 4], [3, 4], [4], []]);  
<immutable digraph with 4 vertices, 6 edges>  
gap> IsTournament(D);  
true  
gap> D := Digraph([[2], [1], [3]]);  
<immutable digraph with 3 vertices, 3 edges>  
gap> IsTournament(D);  
false  
gap> D := CycleDigraph(IsMutableDigraph, 3);  
<mutable digraph with 3 vertices, 3 edges>  
gap> IsTournament(D);  
true  
gap> DigraphRemoveEdge(D, 1, 2);  
<mutable digraph with 3 vertices, 2 edges>  
gap> IsTournament(D);  
false
```

6.2.16  IsTransitiveDigraph

▷ IsTransitiveDigraph(digraph)  

Returns: true or false.

This property is true if the digraph digraph is transitive, and false if it is not. A digraph is transitive if whenever \([ i, j ]\) and \([ j, k ]\) are edges of the digraph, then \([ i, k ]\) is also an edge of the digraph.

Let \(n\) be the number of vertices of an arbitrary digraph, and let \(m\) be the number of edges. For general digraphs, the methods used for this property use a version of the Floyd-Warshall algorithm, and have complexity \(O(n^3)\). However for digraphs which are topologically sortable [DigraphTopologicalSort (5.1.8)], then methods with complexity \(O(m + n + m \cdot n)\) will be used when appropriate.

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

Example

```
gap> D := Digraph([[1, 2], [3], [3]]);  
<immutable digraph with 3 vertices, 4 edges>
Digraphs

6.3 Edge Weights

6.3.1 EdgeWeights

EdgeWeights(digraph)  (attribute)

EdgeWeightsMutableCopy(digraph)  (operation)

Returns: A list of lists of integers, floats or rationals.

EdgeWeights returns the list of lists of edge weights of the edges of the digraph digraph.

More specifically, weights[i][j] is the weight given to the jth edge from vertex i, according to the ordering of edges given by OutNeighbours(digraph)[i].

The function EdgeWeights returns an immutable list of immutable lists, whereas the function EdgeWeightsMutableCopy returns a copy of EdgeWeights which is a mutable list of mutable lists.

The edge weights of a digraph cannot be computed and must be set either using SetEdgeWeights or EdgeWeightedDigraph (6.3.2).

Example

gap> gr := EdgeWeightedDigraph([[2], [3], [1]], [[5], [10], [15]]);  
<immutable digraph with 3 vertices, 3 edges>

gap> EdgeWeights(gr);
[ [ 5 ], [ 10 ], [ 15 ] ]

gap> a := EdgeWeightsMutableCopy(gr);
[ [ 5 ], [ 10 ], [ 15 ] ]

gap> a[1][1] := 100;
100

gap> a;
[ [ 100 ], [ 10 ], [ 15 ] ]

gap> b := EdgeWeights(gr);
[ [ 5 ], [ 10 ], [ 15 ] ]

gap> b[1][1] := 534;
Error, List Assignment: <list> must be a mutable list

6.3.2 EdgeWeightedDigraph

EdgeWeightedDigraph(digraph, weights)  (function)

Returns: A digraph or fail

The argument digraph may be a digraph or a list of lists of integers, floats or rationals.
weights must be a list of lists of integers, floats or rationals of an equal size and shape to
OutNeighbours(digraph), otherwise it will fail.
This will create a digraph and set the EdgeWeights to weights.
See EdgeWeights (6.3.1).

Example

```gap
gap> g := EdgeWeightedDigraph(Digraph([[2], [1]]), [[5], [15]]);
<immutable digraph with 2 vertices, 2 edges>
gap> g := EdgeWeightedDigraph([[2], [1]], [[5], [15]]);
<immutable digraph with 2 vertices, 2 edges>
gap> EdgeWeights(g);
[ [ 5 ], [ 15 ] ]
```

6.4 Orders

6.4.1 IsPreorderDigraph

▷ IsPreorderDigraph(digraph)
▷ IsQuasiorderDigraph(digraph)

Returns: true or false.

A digraph is a preorder digraph if and only if the digraph satisfies both IsReflexiveDigraph (6.2.13) and IsTransitiveDigraph (6.2.16). A preorder digraph (or quasiorder digraph) digraph corresponds to the preorder relation \( \leq \) defined by \( x \leq y \) if and only if \([x, y]\) is an edge of digraph.

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

Example

```gap
gap> D := Digraph([[1], [2, 3], [2, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsPreorderDigraph(D);
true
gap> D := Digraph([[1 .. 4], [1 .. 4], [1 .. 4], [1 .. 4]]);
<immutable digraph with 4 vertices, 16 edges>
gap> IsPreorderDigraph(D);
true
gap> D := Digraph([[2], [3], [4], [5], [1]]);
<immutable digraph with 5 vertices, 5 edges>
gap> IsPreorderDigraph(D);
false
gap> D := Digraph([[1], [1, 2], [2, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsQuasiorderDigraph(D);
false
```

6.4.2 IsPartialOrderDigraph

▷ IsPartialOrderDigraph(digraph)

Returns: true or false.

A digraph is a partial order digraph if and only if the digraph satisfies all of IsReflexiveDigraph (6.2.13), IsAntisymmetricDigraph (6.2.2) and IsTransitiveDigraph (6.2.16). A partial order
digraph corresponds to the partial order relation $\leq$ defined by $x \leq y$ if and only if $[x, y]$ is an edge of digraph.

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

\[
gap> D := \text{Digraph}([[1, 3], [2, 3], [3]]); \\
<\text{immutable digraph with 3 vertices, 5 edges}> \\
gap> \text{IsPartialOrderDigraph}(D); \\
true \\
gap> D := \text{CycleDigraph}(5); \\
<\text{immutable cycle digraph with 5 vertices}> \\
gap> \text{IsPartialOrderDigraph}(D); \\
false \\
gap> D := \text{Digraph}([[1, 1], [1, 1, 2], [3], [3, 3, 4, 4]]); \\
<\text{immutable multidigraph with 4 vertices, 10 edges}> \\
gap> \text{IsPartialOrderDigraph}(D); \\
true
\]

6.4.3 IsMeetSemilatticeDigraph

\[\text{IsMeetSemilatticeDigraph(digraph)}\] (property)
\[\text{IsJoinSemilatticeDigraph(digraph)}\] (property)
\[\text{IsLatticeDigraph(digraph)}\] (property)

Returns: true or false.

IsMeetSemilatticeDigraph returns true if the digraph digraph is a meet semilattice; IsJoinSemilatticeDigraph returns true if the digraph digraph is a join semilattice; and IsLatticeDigraph returns true if the digraph digraph is both a meet and a join semilattice.

For a partial order digraph IsPartialOrderDigraph (6.4.2) the corresponding partial order is the relation $\leq$, defined by $x \leq y$ if and only if $[x, y]$ is an edge. A digraph is a meet semilattice if it is a partial order and every pair of vertices has a greatest lower bound (meet) with respect to the aforementioned relation. A join semilattice is a partial order where every pair of vertices has a least upper bound (join) with respect to the relation.

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

\[
\text{Example} \\
gap> D := \text{Digraph}([[1, 3], [2, 3], [3]]); \\
<\text{immutable digraph with 3 vertices, 5 edges}> \\
gap> \text{IsMeetSemilatticeDigraph}(D); \\
false \\
gap> \text{IsJoinSemilatticeDigraph}(D); \\
true \\
gap> \text{IsLatticeDigraph}(D); \\
false \\
gap> D := \text{Digraph}([[1], [2], [1 .. 3]]); \\
<\text{immutable digraph with 3 vertices, 5 edges}> \\
gap> \text{IsJoinSemilatticeDigraph}(D); \\
false \\
gap> \text{IsMeetSemilatticeDigraph}(D); \\
true \\
gap> \text{IsLatticeDigraph}(D); \\
false
\]
false

\texttt{gap} \> \texttt{D := Digraph([[1 .. 4], [2, 4], [3, 4], [4]]);}
\texttt{<immutable digraph with 4 vertices, 9 edges>}
\texttt{gap} \> \texttt{IsMeetSemilatticeDigraph(D);} 
\texttt{true}
\texttt{gap} \> \texttt{IsJoinSemilatticeDigraph(D);} 
\texttt{true}
\texttt{gap} \> \texttt{IsLatticeDigraph(D);} 
\texttt{true}

6.4.4 DigraphMeetTable

\texttt{DigraphMeetTable} \hspace{1em} (attribute)
\texttt{DigraphJoinTable} \hspace{1em} (attribute)

\textbf{Returns:} A matrix or \texttt{fail}.

\texttt{DigraphMeetTable} returns the \textit{meet table} of \texttt{digraph} if \texttt{digraph} is a meet semilattice digraph \texttt{IsMeetSemilatticeDigraph} (6.4.3). Similarly, \texttt{DigraphJoinTable} returns the \textit{join table} of \texttt{digraph} if \texttt{digraph} is a join semilattice digraph \texttt{IsJoinSemilatticeDigraph} (6.4.3).

When \texttt{digraph} is a meet semilattice digraph with \textit{n} vertices, the \textit{meet table} of \texttt{digraph} is the matrix \texttt{A} such that the \((i,j)\) entry of \texttt{A} is the meet of vertices \(i\) and \(j\).

Similarly, when \texttt{digraph} is a join semilattice digraph with \textit{n} vertices, the \textit{join table} of \texttt{digraph} is the matrix \texttt{A} such that the \((i,j)\) entry of \texttt{A} is the join of vertices \(i\) and \(j\). Otherwise, each function returns \texttt{fail}.

\textbf{Example}

\texttt{gap} \> \texttt{D := Digraph([[1, 2, 3, 4], [2, 4], [3, 4], [4]]);} 
\texttt{<immutable digraph with 4 vertices, 9 edges>}
\texttt{gap} \> \texttt{IsJoinSemilatticeDigraph(D);} 
\texttt{true}
\texttt{gap} \> \texttt{Display(DigraphJoinTable(D));} 
\texttt{[ [ 1, 2, 3, 4 ],} 
\texttt{ [ 2, 2, 4, 4 ],} 
\texttt{ [ 3, 4, 3, 4 ],} 
\texttt{ [ 4, 4, 4, 4 ] ]}
\texttt{gap} \> \texttt{D := CycleDigraph(5);} 
\texttt{<immutable cycle digraph with 5 vertices>}
\texttt{gap} \> \texttt{DigraphJoinTable(D);} 
\texttt{fail}

6.4.5 IsUpperSemimodularDigraph

\texttt{IsUpperSemimodularDigraph(D)} \hspace{1em} (property)
\texttt{IsLowerSemimodularDigraph(D)} \hspace{1em} (property)

\textbf{Returns:} true or \texttt{false}.

\texttt{IsUpperSemimodularDigraph} returns \texttt{true} if the digraph \texttt{D} represents an upper semimodular lattice and \texttt{false} if it does not. Similarly, \texttt{IsLowerSemimodularDigraph} returns \texttt{true} if \texttt{D} represents a lower semimodular lattice and \texttt{false} if it does not.

In a lattice we say that a vertex \(a\) \textit{covers} a vertex \(b\) if \(a\) is greater than \(b\), and there are no further vertices between \(a\) and \(b\). A lattice is \textit{upper semimodular} if whenever the meet of \(a\) and \(b\) is covered by \(a\), the join of \(a\) and \(b\) covers \(b\). Lower semimodularity is defined analogously.
See also IsLatticeDigraph (6.4.3), NonUpperSemimodularPair (5.3.2), and NonLowerSemimodularPair (5.3.2). If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

Example

```gap
d := DigraphFromDigraph6String("&M~sc'LYUZO__KIBboC_@h?U_?_GL?A_?c");
<immutable digraph with 14 vertices, 66 edges>
d := DigraphReflexiveTransitiveClosure(d);
<immutable preorder digraph with 14 vertices, 66 edges>
d := DigraphReflexiveTransitiveClosure(d);
true
```

### 6.4.6 IsDistributiveLatticeDigraph

▷ `IsDistributiveLatticeDigraph(digraph)` (property)

**Returns:** true or false.

IsDistributiveLatticeDigraph returns true if the digraph `digraph` is a *distributive lattice digraph*.

A *distributive lattice digraph* is a *lattice digraph* (IsLatticeDigraph (6.4.3)) which is distributive. That is to say, the functions PartialOrderDigraphMeetOfVertices (5.3.1) and PartialOrderDigraphJoinOfVertices (5.3.1) distribute over each other.

Equivalently, a *distributive lattice digraph* is a *lattice digraph* in which the lattice digraphs representing `M_3` and `N_5` are not embeddable as lattices (see https://en.wikipedia.org/wiki/Distributive_lattice and IsLatticeEmbedding (7.3.20)).

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

Example

```gap
d := Digraph([[2, 3], [4], [4], [3]]);
<immutable digraph with 4 vertices, 4 edges>
d := DigraphReflexiveTransitiveClosure(d);
<immutable preorder digraph with 4 vertices, 9 edges>
d := DigraphReflexiveTransitiveClosure(d);
true
```

### 6.5 Regularity

#### 6.5.1 IsInRegularDigraph

▷ `IsInRegularDigraph(digraph)` (property)

**Returns:** true or false.

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

Example

```gap
d := Digraph([[2, 3], [4], [4], [3]]);
<immutable digraph with 5 vertices, 5 edges>
d := DigraphReflexiveTransitiveClosure(d);
<immutable preorder digraph with 5 vertices, 13 edges>
d := DigraphReflexiveTransitiveClosure(d);
false```
This property is true if there is an integer \( n \) such that for every vertex \( v \) of digraph \( digraph \) there are exactly \( n \) edges terminating in \( v \). See also \texttt{IsOutRegularDigraph} (6.5.2) and \texttt{IsRegularDigraph} (6.5.3).

If the argument \( digraph \) is mutable, then the return value of this property is recomputed every time it is called.

```gap
gap> IsInRegularDigraph(CompleteDigraph(4));
true
gap> IsInRegularDigraph(ChainDigraph(4));
false
```

### 6.5.2 IsOutRegularDigraph

\texttt{IsOutRegularDigraph(digraph)} (property)

\textbf{Returns:} true or false.

This property is true if there is an integer \( n \) such that for every vertex \( v \) of digraph \( digraph \) there are exactly \( n \) edges starting at \( v \).

See also \texttt{IsInRegularDigraph} (6.5.1) and \texttt{IsRegularDigraph} (6.5.3).

If the argument \( digraph \) is mutable, then the return value of this property is recomputed every time it is called.

```gap
gap> IsOutRegularDigraph(CompleteDigraph(4));
true
gap> IsOutRegularDigraph(ChainDigraph(4));
false
```

### 6.5.3 IsRegularDigraph

\texttt{IsRegularDigraph(digraph)} (property)

\textbf{Returns:} true or false.

This property is true if there is an integer \( n \) such that for every vertex \( v \) of digraph \( digraph \) there are exactly \( n \) edges starting and terminating at \( v \). In other words, the property is true if \( digraph \) is both in-regular and out-regular. See also \texttt{IsInRegularDigraph} (6.5.1) and \texttt{IsOutRegularDigraph} (6.5.2).

If the argument \( digraph \) is mutable, then the return value of this property is recomputed every time it is called.

```gap
gap> IsRegularDigraph(CompleteDigraph(4));
true
gap> IsRegularDigraph(ChainDigraph(4));
false
```

### 6.5.4 IsDistanceRegularDigraph

\texttt{IsDistanceRegularDigraph(digraph)} (property)

\textbf{Returns:} true or false.

If \( digraph \) is a connected symmetric graph, this property returns true if for any two vertices \( u \) and \( v \) of \( digraph \) and any two integers \( i \) and \( j \) between 0 and the diameter of \( digraph \), the number
of vertices at distance $i$ from $u$ and distance $j$ from $v$ depends only on $i$, $j$, and the distance between vertices $u$ and $v$.

Alternatively, a distance regular graph is a graph for which there exist integers $b_i$, $c_i$, and $i$ such that for any two vertices $u$, $v$ in $digraph$ which are distance $i$ apart, there are exactly $b_i$ neighbors of $v$ which are at distance $i-1$ away from $u$, and $c_i$ neighbors of $v$ which are at distance $i+1$ away from $u$. This definition is used to check whether $digraph$ is distance regular.

If the argument $digraph$ is mutable, then the return value of this property is recomputed every time it is called.

Example

```gap
gap> D := DigraphSymmetricClosure(ChainDigraph(5));;
gap> IsDistanceRegularDigraph(D);
false

gap> D := Digraph([[2, 3, 4], [1, 3, 4], [1, 2, 4], [1, 2, 3]]);
<immutable digraph with 4 vertices, 12 edges>
gap> IsDistanceRegularDigraph(D);
true
```

### 6.6 Connectivity and cycles

#### 6.6.1 IsAcyclicDigraph

```gap
gap> IsAcyclicDigraph(digraph) # (property)

Returns: true or false.

This property is true if the digraph $digraph$ is acyclic, and false if it is not. A digraph is acyclic if every directed cycle on the digraph is trivial. See Section 1.1.1 for the definition of a directed cycle, and of a trivial directed cycle.

The method used in this operation has complexity $O(m+n)$ where $m$ is the number of edges (counting multiple edges as one) and $n$ is the number of vertices in the digraph.

If the argument $digraph$ is mutable, then the return value of this property is recomputed every time it is called.

Example

```gap
gap> Petersen := Graph(SymmetricGroup(5), [[1, 2]], OnSets,
> function(x, y)
> return IsEmpty(Intersection(x, y));
> end);
<immutable digraph with 10 vertices, 30 edges>
gap> D := Digraph(Petersen);
<immutable digraph with 10 vertices, 30 edges>
gap> IsAcyclicDigraph(D);
false

gap> D := DigraphFromDiSparse6String(".b_OGCIDBaPGkULEbQHCeRIdrHcuZMfryDaPhTi|zF");
<immutable digraph with 35 vertices, 34 edges>
gap> IsAcyclicDigraph(D);
true

gap> IsAcyclicDigraph(ChainDigraph(10));
true

gap> D := CompleteDigraph(IsMutableDigraph, 4);
<mutable digraph with 4 vertices, 12 edges>
```

```
Digraphs

6.6.2 IsChainDigraph

\[ \text{IsChainDigraph}(\text{digraph}) \quad \text{(property)} \]

**Returns:** true or false.

\( \text{IsChainDigraph} \) returns true if the digraph \( \text{digraph} \) is isomorphic to the chain digraph with the same number of vertices as \( \text{digraph} \), and false if it is not; see \( \text{ChainDigraph} \) (3.5.9).

A digraph is a *chain* if and only if it is a directed tree, in which every vertex has out degree at most one; see \( \text{IsDirectedTree} \) (6.6.8) and \( \text{OutDegrees} \) (5.2.8).

If the argument \( \text{digraph} \) is mutable, then the return value of this property is recomputed every time it is called.

Example

\[
\begin{align*}
gap & > \ \text{D} := \text{Digraph}([[1, 3], [2, 3], [3]]); \\
& <\text{immutable digraph with 3 vertices, 5 edges}> \\
& \text{gap} \> \text{IsChainDigraph}(\text{D}); \\
& \text{false} \\
& \text{gap} \> \text{D} := \text{ChainDigraph}(5); \\
& <\text{immutable chain digraph with 5 vertices}> \\
& \text{gap} \> \text{IsChainDigraph}(\text{D}); \\
& \text{true} \\
& \text{gap} \> \text{D} := \text{DigraphReverse}(\text{D}); \\
& <\text{immutable digraph with 5 vertices, 4 edges}> \\
& \text{gap} \> \text{IsChainDigraph}(\text{D}); \\
& \text{true} \\
& \text{gap} \> \text{D} := \text{ChainDigraph}(\text{IsMutableDigraph, 5}); \\
& <\text{mutable digraph with 5 vertices, 4 edges}> \\
& \text{gap} \> \text{IsChainDigraph}(\text{D}); \\
& \text{true} \\
& \text{gap} \> \text{DigraphReverse}(\text{D}); \\
& <\text{mutable digraph with 5 vertices, 4 edges}> \\
& \text{gap} \> \text{IsChainDigraph}(\text{D}); \\
& \text{true}
\end{align*}
\]

6.6.3 IsConnectedDigraph

\[ \text{IsConnectedDigraph}(\text{digraph}) \quad \text{(property)} \]

**Returns:** true or false.

This property is true if the digraph \( \text{digraph} \) is weakly connected and false if it is not. A digraph \( \text{digraph} \) is *weakly connected* if it is possible to travel from any vertex to any other vertex by traversing edges in either direction (possibly against the orientation of some of them).

The method used in this function has complexity \( O(m) \) if the digraph’s \( \text{DigraphSource} \) (5.2.5) attribute is set, otherwise it has complexity \( O(m + n) \) (where \( m \) is the number of edges and \( n \) is the number of vertices of the digraph).

If the argument \( \text{digraph} \) is mutable, then the return value of this property is recomputed every time it is called.
6.6.4 IsBiconnectedDigraph

▷ IsBiconnectedDigraph(digraph)

(property)

Returns: true or false.

A connected digraph is biconnected if it is still connected (in the sense of IsConnectedDigraph (6.6.3)) when any vertex is removed. If $D$ has at least 3 vertices, then IsBiconnectedDigraph implies IsBridgelessDigraph (6.6.5); see ArticulationPoints (5.4.14) or Bridges (5.4.15) for a more detailed explanation.

IsBiconnectedDigraph returns true if the digraph $digraph$ is biconnected, and false if it is not. In particular, IsBiconnectedDigraph returns false if $digraph$ is not connected.

Multiple edges are ignored by this method.

The method used in this operation has complexity $O(m+n)$ where $m$ is the number of edges and $n$ is the number of vertices in the digraph.

See also Bridges (5.4.15), ArticulationPoints (5.4.14), and IsBridgelessDigraph (6.6.5).

If the argument $digraph$ is mutable, then the return value of this property is recomputed every time it is called.

Example

```gap
gap> D := Digraph([[2], [3], []]);;
gap> IsConnectedDigraph(D);
true
gap> D := Digraph([[1, 3], [4], [3], []]);;
gap> IsConnectedDigraph(D);
false
gap> D := Digraph(IsMutableDigraph, [[2], [3], []]);;
gap> IsConnectedDigraph(D);
true
gap> D := Digraph(IsMutableDigraph, [[1, 3], [4], [3], []]);;
gap> IsConnectedDigraph(D);
false
```

6.6.5 IsBridgelessDigraph

▷ IsBridgelessDigraph(digraph)

(property)

Returns: true or false.

A connected digraph is bridgeless if it is still connected (in the sense of IsConnectedDigraph (6.6.3)) when any edge is removed. If $digraph$ has at least 3 vertices, then IsBiconnectedDigraph
Digraphs

(6.6.4) implies IsBridgelessDigraph; see ArticulationPoints (5.4.14) or Bridges (5.4.15) for a more detailed explanation.

IsBridgelessDigraph returns true if the digraph digraph is bridgeless, and false if it is not. In particular, IsBridgelessDigraph returns false if digraph is not connected.

Multiple edges are ignored by this method.

The method used in this operation has complexity \( O(m + n) \) where \( m \) is the number of edges and \( n \) is the number of vertices in the digraph.

See also Bridges (5.4.15), ArticulationPoints (5.4.14), and IsBiconnectedDigraph (6.6.4).

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

Example

```
gap> IsBridgelessDigraph(Digraph([[1, 3], [2, 3], [3]]));
false
gap> IsBridgelessDigraph(CycleDigraph(5));
true
gap> D := Digraph([[1, 1], [1, 1, 2], [3], [3, 3, 4, 4]]);
gap> IsBridgelessDigraph(D);
false
gap> D := CompleteBipartiteDigraph(IsMutableDigraph, 5, 4);
<mutable digraph with 9 vertices, 40 edges>
 gap> IsBridgelessDigraph(D);
true
gap> D := Digraph([[2, 5], [1, 3, 4, 5], [2, 4], [2, 3], [1, 2]]);
<immutable digraph with 5 vertices, 12 edges>
 gap> IsBridgelessDigraph(D);
true
gap> D := Digraph([[2], [3], [4], [2]]);
<immutable digraph with 4 vertices, 4 edges>
 gap> IsBridgelessDigraph(D);
false
gap> IsBiconnectedDigraph(D);
false
gap> IsBridgelessDigraph(ChainDigraph(2));
false
gap> IsBiconnectedDigraph(ChainDigraph(2));
true
```

6.6.6 IsStronglyConnectedDigraph

▷ IsStronglyConnectedDigraph(digraph)  

Returns: true or false.  

This property is true if the digraph digraph is strongly connected and false if it is not.

A digraph digraph is strongly connected if there is a directed path from every vertex to every other vertex. See Section 1.1.1 for the definition of a directed path.

The method used in this operation is based on Gabow’s Algorithm [Gab00] and has complexity \( O(m + n) \), where \( m \) is the number of edges (counting multiple edges as one) and \( n \) is the number of vertices in the digraph.
If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```gap
gap> D := CycleDigraph(250000);
<immutable cycle digraph with 250000 vertices>
gap> IsStronglyConnectedDigraph(D);
true
gap> D := DigraphRemoveEdges(D, [[250000, 1]]);
<immutable digraph with 250000 vertices, 249999 edges>
gap> IsStronglyConnectedDigraph(D);
false
gap> D := CycleDigraph(IsMutableDigraph, 250000);
<mutable digraph with 250000 vertices, 250000 edges>
gap> IsStronglyConnectedDigraph(D);
true
gap> DigraphRemoveEdge(D, [250000, 1]);
<mutable digraph with 250000 vertices, 249999 edges>
gap> IsStronglyConnectedDigraph(D);
false
```

### 6.6.7 `IsAperiodicDigraph`

▶ `IsAperiodicDigraph(digraph)`  

**Returns:** true or false.

This property is true if the digraph `digraph` is aperiodic, i.e. if its `DigraphPeriod (5.4.17)` is equal to 1. Otherwise, the property is false.

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```gap
gap> D := Digraph([[6], [1], [2], [3], [4, 4], [5]]);
<immutable multidigraph with 6 vertices, 7 edges>
gap> IsAperiodicDigraph(D);
false
gap> D := Digraph([[2], [3, 5], [4], [5], [1, 2]]);
<immutable digraph with 5 vertices, 7 edges>
gap> IsAperiodicDigraph(D);
true
gap> D := Digraph(IsMutableDigraph, [[2], [3, 5], [4], [5], [1, 2]]);
<mutable digraph with 5 vertices, 7 edges>
gap> IsAperiodicDigraph(D);
true
```

### 6.6.8 `IsDirectedTree`

▶ `IsDirectedTree(digraph)`  

**Returns:** true or false.

Returns true if the digraph `digraph` is a directed tree, and false if it is not.

A directed tree is an acyclic digraph with precisely 1 source, such that no two vertices share an out-neighbour. Note that the empty digraph with zero vertices is not considered to be a directed tree, because it has no source.

```gap
gap> D := Digraph([[6], [1], [2], [3], [4, 4], [5]]);
<immutable multidigraph with 6 vertices, 7 edges>
gap> IsDirectedTree(D);
false
gap> D := Digraph([[2], [3, 5], [4], [5], [1, 2]]);
<immutable digraph with 5 vertices, 7 edges>
gap> IsDirectedTree(D);
true
```
See also DigraphSources (5.1.7).

If the argument \textit{digraph} is mutable, then the return value of this property is recomputed every time it is called.

\begin{verbatim}
gap> D := Digraph([[0], [2]]);
<immutable digraph with 2 vertices, 1 edge>
gap> IsDirectedTree(D);
false
gap> D := Digraph([[3], [3], []]);
<immutable digraph with 3 vertices, 2 edges>
gap> IsDirectedTree(D);
false
gap> D := Digraph([[2], [3], []]);
<immutable digraph with 3 vertices, 2 edges>
gap> IsDirectedTree(D);
true
gap> D := Digraph([[2, 3], [6], [4, 5], [], [], []]);
<immutable digraph with 6 vertices, 5 edges>
gap> IsDirectedTree(D);
true
\end{verbatim}

\section*{6.6.9 IsUndirectedTree}

▷ IsUndirectedTree\((\text{digraph})\) (property)
▷ IsUndirectedForest\((\text{digraph})\) (property)

\textbf{Returns:} true or false.

The property IsUndirectedTree returns true if the digraph \textit{digraph} is an undirected tree, and the property IsUndirectedForest returns true if \textit{digraph} is an undirected forest; otherwise, these properties return false.

An \textit{undirected tree} is a symmetric digraph without loops, in which for any pair of distinct vertices \(u\) and \(v\), there is exactly one directed path from \(u\) to \(v\). See IsSymmetricDigraph (6.2.14) and DigraphHasLoops (6.2.1), and see Section 1.1.1 for the definition of directed path. This definition implies that an undirected tree has no multiple edges.

An \textit{undirected forest} is a digraph, each of whose connected components is an undirected tree. In other words, an undirected forest is isomorphic to a disjoint union of undirected trees. See DigraphConnectedComponents (5.4.9) and DigraphDisjointUnion (3.3.28). In particular, every undirected tree is an undirected forest.

Please note that the digraph with zero vertices is considered to be neither an undirected tree nor an undirected forest.

If the argument \textit{digraph} is mutable, then the return value of this property is recomputed every time it is called.

\begin{verbatim}
\textbf{Example}
\begin{verbatim}
gap> D := Digraph([[3], [3], [1, 2]]);
<immutable digraph with 3 vertices, 4 edges>
gap> IsUndirectedTree(D);
true
gap> IsSymmetricDigraph(D) and not DigraphHasLoops(D);
true
gap> D := Digraph([[3], [5], [1, 4], [3], [2]]);
\end{verbatim}
\end{verbatim}

\end{verbatim}
<immutable digraph with 5 vertices, 6 edges>
gap> IsConnectedDigraph(D);
false
gap> IsUndirectedTree(D);
false
gap> IsUndirectedForest(D);
true

<immutable digraph with 3 vertices, 4 edges>
gap> IsUndirectedTree(D) or IsUndirectedForest(D);
false

6.6.10  IsEulerianDigraph

▷ IsEulerianDigraph(digraph) (property)

Returns: true or false.

This property returns true if the digraph digraph is Eulerian.

A connected digraph is called Eulerian if there exists a directed circuit on the digraph which includes every edge exactly once. See Section 1.1.1 for the definition of a directed circuit. Note that the empty digraph with at most one vertex is considered to be Eulerian.

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

Example

gap> D := Digraph([[1], [2]]);
<immutable empty digraph with 1 vertex>

gap> IsEulerianDigraph(D);
true

gap> D := Digraph([[2], []]);
<immutable digraph with 2 vertices, 1 edge>

gap> IsEulerianDigraph(D);
false

6.6.11  IsHamiltonianDigraph

▷ IsHamiltonianDigraph(digraph) (property)

Returns: true or false.

If digraph is Hamiltonian, then this property returns true, and false if it is not.

A digraph with n vertices is Hamiltonian if it has a directed cycle of length n. See Section 1.1.1 for the definition of a directed cycle. Note the empty digraphs on 0 and 1 vertices are considered to be Hamiltonian.
The method used in this operation has the worst case complexity as DigraphMonomorphism (7.3.4).

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

```
gap> g := Digraph([[]]);
<immutable empty digraph with 1 vertex>
gap> IsHamiltonianDigraph(g);
true
gap> g := Digraph([[2], [1]]);
<immutable digraph with 2 vertices, 2 edges>
gap> IsHamiltonianDigraph(g);
true
gap> g := Digraph([[3], [], [2]]);
<immutable digraph with 3 vertices, 2 edges>
gap> IsHamiltonianDigraph(g);
false
gap> g := Digraph([[2], [3], [1]]);
<immutable digraph with 3 vertices, 3 edges>
gap> IsHamiltonianDigraph(g);
true
```

### 6.6.12 IsCycleDigraph

▷ IsCycleDigraph(digraph) (property)

Returns: true or false.

IsCycleDigraph returns true if the digraph digraph is isomorphic to the cycle digraph with the same number of vertices as digraph, and false if it is not; see CycleDigraph (3.5.14).

A digraph is a cycle if and only if it is strongly connected and has the same number of edges as vertices.

If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

```
gap> D := Digraph([[1, 3], [2, 3], [3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsCycleDigraph(D);
false
gap> D := CycleDigraph(5);
<immutable cycle digraph with 5 vertices>
gap> IsCycleDigraph(D);
true
gap> D := OnDigraphs(D, (1, 2, 3));
<immutable digraph with 5 vertices, 5 edges>
gap> D = CycleDigraph(5);
false
gap> IsCycleDigraph(D);
true
```
6.7 Planarity

6.7.1 IsPlanarDigraph

▷ IsPlanarDigraph(digraph) (property)

Returns: true or false.

A planar digraph is a digraph that can be embedded in the plane in such a way that its edges do not intersect. A digraph is planar if and only if it does not have a subdigraph that is homeomorphic to either the complete graph on 5 vertices or the complete bipartite graph with vertex sets of sizes 3 and 3.

IsPlanarDigraph returns true if the digraph digraph is planar and false if it is not. The directions and multiplicities of any edges in digraph are ignored by IsPlanarDigraph.

See also IsOuterPlanarDigraph (6.7.2). This method uses the reference implementation in edge-addition-planarity-suite by John Boyer of the algorithms described in [BM06].

Example

```gap
gap> IsPlanarDigraph(CompleteDigraph(4));
true
gap> IsPlanarDigraph(CompleteDigraph(5));
false
gap> IsPlanarDigraph(CompleteBipartiteDigraph(2, 3));
true
gap> IsPlanarDigraph(CompleteBipartiteDigraph(3, 3));
false
gap> IsPlanarDigraph(CompleteDigraph(IsMutableDigraph, 4));
true
gap> IsPlanarDigraph(CompleteDigraph(IsMutableDigraph, 5));
false
gap> IsPlanarDigraph(CompleteBipartiteDigraph(IsMutableDigraph, 2, 3));
true
gap> IsPlanarDigraph(CompleteBipartiteDigraph(IsMutableDigraph, 3, 3));
false
```

6.7.2 IsOuterPlanarDigraph

▷ IsOuterPlanarDigraph(digraph) (property)

Returns: true or false.

An outer planar digraph is a digraph that can be embedded in the plane in such a way that its edges do not intersect, and all vertices belong to the unbounded face of the embedding. A digraph is outer planar if and only if it does not have a subdigraph that is homeomorphic to either the complete graph on 4 vertices or the complete bipartite graph with vertex sets of sizes 2 and 3.

IsOuterPlanarDigraph returns true if the digraph digraph is outer planar and false if it is not. The directions and multiplicities of any edges in digraph are ignored by IsPlanarDigraph.

See also IsPlanarDigraph (6.7.1). This method uses the reference implementation in edge-addition-planarity-suite by John Boyer of the algorithms described in [BM06].

Example

```gap
gap> IsOuterPlanarDigraph(CompleteDigraph(4));
false
gap> IsOuterPlanarDigraph(CompleteDigraph(5));
false
```
Digraphs

6.8 Homomorphisms and transformations

6.8.1 IsDigraphCore

▷ IsDigraphCore(digraph) (property)

Returns: true or false.
This property returns true if digraph is a core, and false if it is not.
A digraph D is a core if and only if it has no proper subdigraphs A such that there exists a homomorphism from D to A. In other words, a digraph D is a core if and only if every endomorphism on D is an automorphism on D.
If the argument digraph is mutable, then the return value of this property is recomputed every time it is called.

Example

gap> D := CompleteDigraph(6);
<immutable complete digraph with 6 vertices>
gap> IsDigraphCore(D);
true

gap> D := DigraphSymmetricClosure(CycleDigraph(6));
<immutable symmetric digraph with 6 vertices, 12 edges>
gap> DigraphHomomorphism(D, CompleteDigraph(2));
Transformation( [ 1, 2, 1, 2, 1, 2 ] )
gap> IsDigraphCore(D);
false

6.8.2 IsEdgeTransitive

▷ IsEdgeTransitive(digraph) (property)

Returns: true or false.
If `digraph` is a digraph without multiple edges, then `IsEdgeTransitive` returns `true` if `digraph` is edge transitive, and `false` otherwise. A digraph is edge transitive if its automorphism group acts transitively on its edges (via the action `OnPairs` (Reference: `OnPairs`)).

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```gap
gap> IsEdgeTransitive(CompleteDigraph(2));
true
gap> IsEdgeTransitive(ChainDigraph(3));
false
gap> IsEdgeTransitive(Digraph([[2], [3, 3, 3], []]));
Error, the argument <D> must be a digraph with no multiple edges,
```

### 6.8.3 `IsVertexTransitive`

▷ `IsVertexTransitive(digraph)`

Returns: true or false.

If `digraph` is a digraph, then `IsVertexTransitive` returns `true` if `digraph` is vertex transitive, and `false` otherwise. A digraph is vertex transitive if its automorphism group acts transitively on its vertices.

If the argument `digraph` is mutable, then the return value of this property is recomputed every time it is called.

```gap
gap> IsVertexTransitive(CompleteDigraph(2));
true
gap> IsVertexTransitive(ChainDigraph(3));
false
```
Chapter 7

Homomorphisms

7.1 Acting on digraphs

7.1.1 OnDigraphs (for a digraph and a perm)

▷ OnDigraphs(digraph, perm)
▷ OnDigraphs(digraph, trans)

Returns: A digraph.

If digraph is a digraph, and the second argument perm is a permutation of the vertices of digraph, then this operation returns a digraph constructed by relabelling the vertices of digraph according to perm. Note that for an automorphism f of a digraph, we have OnDigraphs(digraph, f) = digraph.

If the second argument is a transformation trans of the vertices of digraph, then this operation returns a digraph constructed by transforming the source and range of each edge according to trans. Thus a vertex which does not appear in the image of trans will be isolated in the returned digraph, and the returned digraph may contain multiple edges, even if digraph does not. If trans is mathematically a permutation, then the result coincides with OnDigraphs(digraph, AsPermutation(trans)).

The DigraphVertexLabels (5.1.10) of digraph will not be retained in the returned digraph.

If digraph belongs to IsMutableDigraph (3.1.2), then relabelling of the vertices is performed directly on digraph. If digraph belongs to IsImmutableDigraph (3.1.3), an immutable copy of digraph with the vertices relabelled is returned.

```gap
gap> D := Digraph([[3], [1, 3, 5], [1], [1, 2, 4], [2, 3, 5]]);
<immutable digraph with 5 vertices, 11 edges>
gap> new := OnDigraphs(D, (1, 2));
<immutable digraph with 5 vertices, 11 edges>
gap> OutNeighbours(new);
[ [ 2, 3, 5 ], [ 3 ], [ 2 ], [ 2, 1, 4 ], [ 1, 3, 5 ] ]
gap> D := Digraph([[2], [], [2]]);
<immutable digraph with 3 vertices, 2 edges>
gap> t := Transformation([1, 2, 1]);
gap> new := OnDigraphs(D, t);
<immutable multidigraph with 3 vertices, 2 edges>
gap> OutNeighbours(new);
[ [ 2, 2 ], [ ], [ ] ]
```
7.1.2 OnMultiDigraphs

▷ OnMultiDigraphs(digraph, pair)  
▷ OnMultiDigraphs(digraph, perm1, perm2)  

Returns: A digraph.

If digraph is a digraph, and pair is a pair consisting of a permutation of the vertices and a permutation of the edges of digraph, then this operation returns a digraph constructed by relabelling the vertices and edges of digraph according to perm[1] and perm[2], respectively.

In its second form, OnMultiDigraphs returns a digraph with vertices and edges permuted by perm1 and perm2, respectively.

Note that OnDigraphs(digraph, perm)=OnMultiDigraphs(digraph, [perm, ()]) where perm is a permutation of the vertices of digraph. If you are only interested in the action of a permutation on the vertices of a digraph, then you can use OnDigraphs instead of OnMultiDigraphs. If digraph belongs to IsMutableDigraph (3.1.2), then relabelling of the vertices is performed directly on digraph. If digraph belongs to IsImmutableDigraph (3.1.3), an immutable copy of digraph with the vertices relabelled is returned.

Example

```gap
gap> D1 := Digraph([ [3, 6, 3], [], [9, 10], [9], [], [10, 4, 10], [], []]);
<immutable multidigraph with 10 vertices, 10 edges>
gap> p := BlissCanonicalLabelling(D1);  
[ (1,7)(3,6)(4,10)(5,9), () ]
gap> D2 := OnMultiDigraphs(D1, p);  
<immutable multidigraph with 10 vertices, 10 edges>
gap> OutNeighbours(D2);  
[ [], [], [ ], [ ], [ ], [ 6 ], [ 6, 3, 6 ], [ 4, 10, 4 ],  
[ 5 ], [ 5, 4 ] ]
```

7.1.3 OnTuplesDigraphs (for a list of digraphs and a perm)

▷ OnTuplesDigraphs(list, perm)  
▷ OnSetsDigraphs(list, perm)  

Returns: A list or set of digraphs.

If list is a list of digraphs, and perm is a permutation of the vertices of the digraphs in list, then OnTuplesDigraphs returns a new list constructed by applying perm via OnDigraphs (7.1.1) to a copy (with the same mutability) of each entry of list in turn.

More precisely, OnTuplesDigraphs(list, perm) is a list of length Length(list), whose i-th entry is OnDigraphs(DigraphCopy(list[i]), perm).

If list is moreover a GAP set (i.e. a duplicate-free sorted list), then OnSetsDigraphs returns the sorted output of OnTuplesDigraphs, which is therefore again a set.

Example

```gap
gap> list := [CycleDigraph(IsImmutableDigraph, 6),  
            DigraphReverse(CycleDigraph(6))];  
[ <mutable digraph with 6 vertices, 6 edges>,
  <mutable digraph with 6 vertices, 6 edges>]
```
<immutable digraph with 6 vertices, 6 edges> ]
gap> p := (1, 6)(2, 5)(3, 4);;
gap> result_tuples := OnTuplesDigraphs(list, p);
[ <mutable digraph with 6 vertices, 6 edges>,
  <immutable digraph with 6 vertices, 6 edges> ]
gap> result_tuples[2] = OnDigraphs(list[2], p);
true
gap> result_tuples = list;
false
gap> result_tuples = Reversed(list);
true
gap> result_sets := OnSetsDigraphs(list, p);
[ <immutable digraph with 6 vertices, 6 edges>,
  <mutable digraph with 6 vertices, 6 edges> ]
gap> result_sets = list;
true

7.2 Isomorphisms and canonical labellings

From version 0.11.0 of Digraphs it is possible to use either bliss or nauty (via NautyTracesInterface) to calculate canonical labellings and automorphism groups of digraphs; see [JK07] and [MP14] for more details about bliss and nauty, respectively.

7.2.1 DigraphsUseNauty

▷ DigraphsUseNauty()

▷ DigraphsUseBliss()

Returns: Nothing.

These functions can be used to specify whether nauty or bliss should be used by default by Digraphs. If NautyTracesInterface is not available, then these functions do nothing. Otherwise, by calling DigraphsUseNauty subsequent computations will default to using nauty rather than bliss, where possible.

You can call these functions at any point in a GAP session, as many times as you like, it is guaranteed that existing digraphs remain valid, and that comparison of existing digraphs and newly created digraphs via IsIsomorphicDigraph (7.2.15), IsIsomorphicDigraph (7.2.16), IsomorphismDigraphs (7.2.17), and IsomorphismDigraphs (7.2.18) are also valid.

It is also possible to compute the automorphism group of a specific digraph using both nauty and bliss using NautyAutomorphismGroup (7.2.4) and BlissAutomorphismGroup (7.2.3), respectively.

7.2.2 AutomorphismGroup (for a digraph)

▷ AutomorphismGroup(digraph)

Returns: A permutation group.

If digraph is a digraph, then this attribute contains the group of automorphisms of digraph. An automorphism of digraph is an isomorphism from digraph to itself. See IsomorphismDigraphs (7.2.17) for more information about isomorphisms of digraphs.

If digraph is not a multidigraph then the automorphism group is returned as a group of permutations on the set of vertices of digraph.
If `digraph` is a multidigraph then the automorphism group is returned as the direct product of a group of permutations on the set of vertices of `digraph` with a group of permutations on the set of edges of `digraph`. These groups can be accessed using Projection (Reference: Projection for a domain and a positive integer) on the returned group.

By default, the automorphism group is found using bliss by Tommi Junttila and Petteri Kaski. If NautyTracesInterface is available, then nauty by Brendan McKay and Adolfo Piperno can be used instead; see BlissAutomorphismGroup (7.2.3), NautyAutomorphismGroup (7.2.4), DigraphsUseBliss (7.2.1), and DigraphsUseNauty (7.2.1).

If the argument `digraph` is mutable, then the return value of this attribute is recomputed every time it is called.

```gap
gap> johnson := DigraphFromGraph6String("E}lw");
<immutable symmetric digraph with 6 vertices, 24 edges>
gap> G := AutomorphismGroup(johnson);
Group([ (3,4), (2,3)(4,5), (1,2)(5,6) ])
gap> cycle := CycleDigraph(9);
<immutable cycle digraph with 9 vertices>
gap> G := AutomorphismGroup(cycle);
Group([ (1,2,3,4,5,6,7,8,9) ])
gap> IsCyclic(G) and Size(G) = 9;
true
```

### 7.2.3 BlissAutomorphismGroup (for a digraph)

- BlissAutomorphismGroup(digraph)  
  (attribute)
- BlissAutomorphismGroup(digraph, vertex_colours)  
  (operation)
- BlissAutomorphismGroup(digraph, vertex_colours, edge_colours)  
  (operation)

**Returns:** A permutation group.

If `digraph` is a digraph, then this attribute contains the group of automorphisms of `digraph` as calculated using bliss by Tommi Junttila and Petteri Kaski.

The attribute AutomorphismGroup (7.2.2) and operation AutomorphismGroup (7.2.5) returns the value of either BlissAutomorphismGroup or NautyAutomorphismGroup (7.2.4). These groups are, of course, equal but their generating sets may differ.

The attribute AutomorphismGroup (7.2.6) returns the value of BlissAutomorphismGroup as it is not implemented for nauty. The requirements for the optional arguments `vertex_colours` and `edge_colours` are documented in AutomorphismGroup (7.2.6). See also DigraphsUseBliss (7.2.1), and DigraphsUseNauty (7.2.1).

If the argument `digraph` is mutable, then the return value of this attribute is recomputed every time it is called.

```gap
gap> G := BlissAutomorphismGroup(JohnsonDigraph(5, 2));;
gap> IsSymmetricGroup(G);
true
gap> Size(G);
120
```
7.2.4 **NautyAutomorphismGroup**

▷ **NautyAutomorphismGroup**(digraph[, vert_colours]) (attribute)

**Returns:** A permutation group.

If `digraph` is a digraph, then this attribute contains the group of automorphisms of `digraph` as calculated using `nauty` by Brendan Mckay and Adolfo Piperno via `NautyTracesInterface`. The attribute `AutomorphismGroup` (7.2.2) and operation `AutomorphismGroup` (7.2.5) returns the value of either `NautyAutomorphismGroup` or `BlissAutomorphismGroup` (7.2.3). These groups are, of course, equal but their generating sets may differ.

See also `DigraphsUseBliss` (7.2.1), and `DigraphsUseNauty` (7.2.1).

If the argument `digraph` is mutable, then the return value of this attribute is recomputed every time it is called.

```
Example
gap> NautyAutomorphismGroup(JohnsonDigraph(5, 2));
Group([ (3,4)(6,7)(8,9), (2,3)(5,6)(9,10), (2,5)(3,6)(4,7),
       (1,2)(6,8)(7,9) ])
```

7.2.5 **AutomorphismGroup** (for a digraph and a homogeneous list)

▷ **AutomorphismGroup**(digraph, vert_colours) (operation)

**Returns:** A permutation group.

This operation computes the automorphism group of a vertex-coloured digraph. A vertex-coloured digraph can be specified by its underlying digraph `digraph` and its colouring `vert_colours`. Let `n` be the number of vertices of `digraph`. The colouring `vert_colours` may have one of the following two forms:

- a list of `n` integers, where `vert_colours[i]` is the colour of vertex `i`, using the colours `[1 .. m]` for some `m <= n`; or

- a list of non-empty disjoint lists whose union is `DigraphVertices(digraph)`, such that `vert_colours[i]` is the list of all vertices with colour `i`.

The automorphism group of a coloured digraph `digraph` with colouring `vert_colours` is the group consisting of its automorphisms; an automorphism of `digraph` is an isomorphism of coloured digraphs from `digraph` to itself. This group is equal to the subgroup of `AutomorphismGroup(digraph)` consisting of those automorphisms that preserve the colouring specified by `vert_colours`. See `AutomorphismGroup` (7.2.2), and see `IsomorphismDigraphs` (7.2.18) for more information about isomorphisms of coloured digraphs.

If `digraph` is not a multidigraph then the automorphism group is returned as a group of permutations on the set of vertices of `digraph`.

If `digraph` is a multidigraph then the automorphism group is returned as the direct product of a group of permutations on the set of vertices of `digraph` with a group of permutations on the set of edges of `digraph`. These groups can be accessed using `Projection` (Reference: Projection for a domain and a positive integer) on the returned group.

By default, the automorphism group is found using `bliss` by Tommi Junttila and Petteri Kaski. If `NautyTracesInterface` is available, then `nauty` by Brendan Mckay and Adolfo Piperno can be used instead; see `BlissAutomorphismGroup` (7.2.3), `NautyAutomorphismGroup` (7.2.4), `DigraphsUseBliss` (7.2.1), and `DigraphsUseNauty` (7.2.1).
Example

```gap
gap> cycle := CycleDigraph(9);
<immutable cycle digraph with 9 vertices>
gap> G := AutomorphismGroup(cycle);
true
gap> IsCyclic(G) and Size(G) = 9;
true
gap> colours := [[1, 4, 7], [2, 5, 8], [3, 6, 9]];

gap> H := AutomorphismGroup(cycle, colours);

gap> Size(H);
3

gap> H = AutomorphismGroup(cycle, [1, 2, 3, 1, 2, 3, 1, 2, 3]);
true

gap> H = SubgroupByProperty(G, p -> OnTuplesSets(colours, p) = colours);
true

gap> IsTrivial(AutomorphismGroup(cycle, [1, 1, 2, 2, 2, 2, 2, 2, 2]));
true
```

7.2.6 AutomorphismGroup (for a digraph, homogeneous list, and list)

▷ AutomorphismGroup(digraph, vert_colours, edge_colours) (operation)

Returns: A permutation group.

This operation computes the automorphism group of a vertex- and/or edge-coloured digraph. A coloured digraph can be specified by its underlying digraph `digraph` and colourings `vert_colours`, `edge_colours`. Let `n` be the number of vertices of `digraph`. The colourings must have the following forms:

- `vert_colours` must be `fail` or a list of `n` integers, where `vert_colours[i]` is the colour of vertex `i`, using the colours `[1 .. m]` for some `m <= n`;

- `edge_colours` must be `fail` or a list of `n` lists of integers of the same shape as `OutNeighbours(digraph)`, where `edge_colours[i][j]` is the colour of the edge `OutNeighbours(digraph)[i][j]`, using the colours `[1 .. k]` for some `k <= n`;

Giving `vert_colours [edge_colours]` as `fail` is equivalent to setting all vertices [edges] to be the same colour.

Unlike AutomorphismGroup (7.2.2), it is possible to obtain the automorphism group of an edge-coloured multidigraph (see IsMultiDigraph (6.2.11)) when no two edges share the same source, range, and colour. The automorphism group of a vertex/edge-coloured digraph `digraph` with colouring `c` is the group consisting of its vertex/edge-colour preserving automorphisms; an automorphism of `digraph` is an isomorphism of vertex/edge-coloured digraphs from `digraph` to itself. This group is equal to the subgroup of AutomorphismGroup(`digraph`) consisting of those automorphisms that preserve the colouring specified by `colours`. See AutomorphismGroup (7.2.2), and see IsomorphismDigraphs (7.2.18) for more information about isomorphisms of coloured digraphs.

If `digraph` is not a multidigraph then the automorphism group is returned as a group of permutations on the set of vertices of `digraph`.

If `digraph` is a multidigraph then the automorphism group is returned as the direct product of a group of permutations on the set of vertices of `digraph` with a group of permutations on the set of edges of `digraph`. These groups can be accessed using Projection (Reference: Projection for a domain and a positive integer) on the returned group.
By default, the automorphism group is found using bliss by Tommi Junttila and Petteri Kaski. If NautyTracesInterface is available, then nauty by Brendan McKay and Adolfo Piperno can be used instead; see BlissAutomorphismGroup (7.2.3), NautyAutomorphismGroup (7.2.4), DigraphsUseBliss (7.2.1), and DigraphsUseNauty (7.2.1).

```gap
gap> cycle := CycleDigraph(12);
<immutable cycle digraph with 12 vertices>
gap> vert_colours := List([1 .. 12], x -> x mod 3 + 1);
gap> edge_colours := List([1 .. 12], x -> [x mod 2 + 1]);
gap> Size(AutomorphismGroup(cycle));
12
gap> Size(AutomorphismGroup(cycle, vert_colours));
4
gap> Size(AutomorphismGroup(cycle, fail, edge_colours));
6
gap> Size(AutomorphismGroup(cycle, vert_colours, edge_colours));
2
gap> IsTrivial(AutomorphismGroup(cycle,
> vert_colours, List([1 .. 12], x -> [x mod 4 + 1])));
true
```

### 7.2.7 BlissCanonicalLabelling (for a digraph)

<table>
<thead>
<tr>
<th>BlissCanonicalLabelling(digraph)</th>
<th>(attribute)</th>
</tr>
</thead>
<tbody>
<tr>
<td>NautyCanonicalLabelling(digraph)</td>
<td>(attribute)</td>
</tr>
</tbody>
</table>

**Returns:** A permutation, or a list of two permutations.

A function \( \rho \) that maps a digraph to a digraph is a **canonical representative map** if the following two conditions hold for all digraphs \( G \) and \( H \):

- \( \rho(G) \) and \( G \) are isomorphic as digraphs; and
- \( \rho(G) = \rho(H) \) if and only if \( G \) and \( H \) are isomorphic as digraphs.

A **canonical labelling** of a digraph \( G \) (under \( \rho \)) is an isomorphism of \( G \) onto its **canonical representative**, \( \rho(G) \). See IsomorphismDigraphs (7.2.17) for more information about isomorphisms of digraphs.

BlissCanonicalLabelling returns a canonical labelling of the digraph \( \text{digraph} \) found using bliss by Tommi Junttila and Petteri Kaski. NautyCanonicalLabelling returns a canonical labelling of the digraph \( \text{digraph} \) found using nauty by Brendan McKay and Adolfo Piperno. Note that the canonical labellings returned by bliss and nauty are not usually the same (and may depend on the version used).

BlissCanonicalLabelling can only be computed if \( \text{digraph} \) has no multiple edges; see IsMultiDigraph (6.2.11).

```gap
gap> digraph1 := DigraphFromDiSparse6String(".ImNS_AiB?qRN");
<immutable digraph with 10 vertices, 8 edges>
gap> BlissCanonicalLabelling(digraph1);
(1,9,5,7)(3,6,4,10)
gap> p := (1, 2, 7, 5)(3, 9)(6, 10, 8);
gap> digraph2 := OnDigraphs(digraph1, p);
```

---

**Example**

```gap
gap> cycle := CycleDigraph(12);
<immutable cycle digraph with 12 vertices>
gap> vert_colours := List([1 .. 12], x -> x mod 3 + 1);
gap> edge_colours := List([1 .. 12], x -> [x mod 2 + 1]);
gap> Size(AutomorphismGroup(cycle));
12
gap> Size(AutomorphismGroup(cycle, vert_colours));
4
gap> Size(AutomorphismGroup(cycle, fail, edge_colours));
6
gap> Size(AutomorphismGroup(cycle, vert_colours, edge_colours));
2
gap> IsTrivial(AutomorphismGroup(cycle,
> vert_colours, List([1 .. 12], x -> [x mod 4 + 1])));
true
```
Digraphs

> immutable digraph with 10 vertices, 8 edges

gap> digraph1 = digraph2;
false

gap> OnDigraphs(digraph1, BlissCanonicalLabelling(digraph1)) =
> OnDigraphs(digraph2, BlissCanonicalLabelling(digraph2));
true

7.2.8 BlissCanonicalLabelling (for a digraph and a list)

▷ BlissCanonicalLabelling(digraph, colours)
▷ NautyCanonicalLabelling(digraph, colours)

Returns: A permutation.

A function $\rho$ that maps a coloured digraph to a coloured digraph is a canonical representative map if the following two conditions hold for all coloured digraphs $G$ and $H$:

- $\rho(G)$ and $G$ are isomorphic as coloured digraphs; and
- $\rho(G) = \rho(H)$ if and only if $G$ and $H$ are isomorphic as coloured digraphs.

A canonical labelling of a coloured digraph $G$ (under $\rho$) is an isomorphism of $G$ onto its canonical representative, $\rho(G)$. See IsomorphismDigraphs (7.2.18) for more information about isomorphisms of coloured digraphs.

A coloured digraph can be specified by its underlying digraph $\text{digraph}$ and its colouring $\text{colours}$. Let $n$ be the number of vertices of $\text{digraph}$. The colouring $\text{colours}$ may have one of the following two forms:

- a list of $n$ integers, where $\text{colours}[i]$ is the colour of vertex $i$, using the colours $[1 .. m]$ for some $m \leq n$; or
- a list of non-empty disjoint lists whose union is $\text{DigraphVertices(digraph)}$, such that $\text{colours}[i]$ is the list of all vertices with colour $i$.

If $\text{digraph}$ and $\text{colours}$ together form a coloured digraph, BlissCanonicalLabelling returns a canonical labelling of the digraph $\text{digraph}$ found using bliss by Tommi Junttila and Petteri Kaski. Similarly, NautyCanonicalLabelling returns a canonical labelling of the digraph $\text{digraph}$ found using nauty by Brendan McKay and Adolfo Piperno. Note that the canonical labellings returned by bliss and nauty are not usually the same (and may depend on the version used).

BlissCanonicalLabelling can only be computed if $\text{digraph}$ has no multiple edges; see IsMultiDigraph (6.2.11). The canonical labelling of $\text{digraph}$ is given as a permutation of its vertices. The canonical representative of digraph can be created from $\text{digraph}$ and its canonical labelling $p$ by using the operation OnDigraphs (7.1.1):

Example

gap> OnDigraphs(digraph, p);

The colouring of the canonical representative can easily be constructed. A vertex $v$ (in $\text{digraph}$) has colour $i$ if and only if the vertex $v^p$ (in the canonical representative) has colour $i$, where $p$ is the permutation of the canonical labelling that acts on the vertices of $\text{digraph}$. In particular, if $\text{colours}$ has the first form that is described above, then the colouring of the canonical representative is given by:
On the other hand, if `colours` has the second form above, then the canonical representative has colouring:

\[
\text{Example} \quad \text{gap} \rightarrow \text{OnTuplesSets(colours, p)};
\]

\[
\text{Example} \quad \text{gap} \rightarrow \text{OnDigraphs(digraph, p)};
\]

\[
\text{Example} \quad \text{gap} \rightarrow \text{List(DigraphVertices(digraph), i -> colours[i / p])};
\]

7.2.9 BlissCanonicalDigraph

\[\text{BlissCanonicalDigraph}(\text{digraph})\]

\[\text{NautyCanonicalDigraph}(\text{digraph})\]

\textbf{Returns:} A digraph.

The attribute BlissCanonicalLabelling returns the canonical representative found by applying BlissCanonicalLabelling (7.2.7). The digraph returned is canonical in the sense that

- BlissCanonicalDigraph(digraph) and digraph are isomorphic as digraphs; and

- If gr is any digraph then BlissCanonicalDigraph(gr) and BlissCanonicalDigraph(digraph) are equal if and only if gr and digraph are isomorphic as digraphs.

Analogously, the attribute NautyCanonicalLabelling returns the canonical representative found by applying NautyCanonicalLabelling (7.2.7).

If the argument digraph is mutable, then the return value of this attribute is recomputed every time it is called.

\[
\text{Example} \quad \text{gap} \rightarrow \text{digraph := Digraph([[1], [2, 3], [3], [1, 2, 3]])};
\]

\[
\text{Example} \quad \text{gap} \rightarrow \text{canon := BlissCanonicalDigraph(digraph)};
\]

\[
\text{Example} \quad \text{gap} \rightarrow \text{OutNeighbours(canon)};
\]
7.2.10 DigraphGroup

▷ DigraphGroup(digraph)

Returns: A permutation group.

If digraph is immutable and was created knowing a subgroup of its automorphism group, then this group is stored in the attribute DigraphGroup. If digraph is mutable, or was not created knowing a subgroup of its automorphism group, then DigraphGroup returns the entire automorphism group of digraph. Note that if digraph is mutable, then the automorphism group is recomputed every time this function is called.

Note that certain other constructor operations such as CayleyDigraph (3.1.12), BipartiteDoubleDigraph (3.3.43), and DoubleDigraph (3.3.42), may not require a group as one of the arguments, but use the standard constructor method using a group, and hence set the DigraphGroup attribute for the resulting digraph.

Example

```gap
gap> n := 4;;
gap> adj := function(x, y)>
> return (((x - y) mod n) = 1) or (((x - y) mod n) = n - 1);
> end;;
gap> group := CyclicGroup(IsPermGroup, n);
Group([ (1,2,3,4) ])
gap> D := Digraph(IsMutableDigraph, group, [1 .. n], \^, adj);
<mutable digraph with 4 vertices, 8 edges>
gap> HasDigraphGroup(D);
false
gap> DigraphGroup(D);
Group([ (2,4), (1,2)(3,4) ])
gap> AutomorphismGroup(D);
Group([ (2,4), (1,2)(3,4) ])
gap> D := Digraph(group, [1 .. n], \^, adj);
<immutable digraph with 4 vertices, 8 edges>
gap> HasDigraphGroup(D);
true
gap> DigraphGroup(D);
Group([ (1,2,3,4) ])
gap> D := DoubleDigraph(D);
<immutable digraph with 8 vertices, 32 edges>
gap> HasDigraphGroup(D);
true
gap> DigraphGroup(D);
Group([ (1,2,3,4)(5,6,7,8), (1,5)(2,6)(3,7)(4,8) ])
gap> AutomorphismGroup(D) =
> Group([ (6,8), (5,7), (4,6), (3,5), (2,4),
> (1,2)(3,4)(5,6)(7,8) ]);
true
gap> D := Digraph([[2, 3], [], []]);
<immutable digraph with 3 vertices, 2 edges>
gap> HasDigraphGroup(D);
false
gap> HasAutomorphismGroup(D);
false
gap> DigraphGroup(D);
Group([ (2,3) ])
```
7.2.11 DigraphOrbits

\[ \text{DigraphOrbits}(\text{digraph}) \]

\textbf{Returns:} An immutable list of lists of integers.

DigraphOrbits returns the orbits of the action of the DigraphGroup (7.2.10) on the set of vertices of\textit{ digraph}.

\begin{verbatim}
gap> G := Group([([2, 3](7, 8, 9), (1, 2, 3)(4, 5, 6)(8, 9))]);;
gap> D := EdgeOrbitsDigraph(G, [1, 2]);
<immutable digraph with 9 vertices, 6 edges>
gap> DigraphOrbits(D);
[ [ 1, 2, 3 ], [ 4, 5, 6 ], [ 7, 8, 9 ] ]
gap> D := DigraphMutableCopy(D);
<mutable digraph with 9 vertices, 6 edges>
gap> DigraphOrbits(D);
[ [ 1, 2, 3 ], [ 4, 5, 6, 7, 8, 9 ] ]
\end{verbatim}

7.2.12 DigraphOrbitReps

\[ \text{DigraphOrbitReps}(\text{digraph}) \]

\textbf{Returns:} An immutable list of integers.

DigraphOrbitReps returns a list of orbit representatives of the action of the DigraphGroup (7.2.10) on the set of vertices of\textit{ digraph}.

\begin{verbatim}
gap> D := CayleyDigraph(AlternatingGroup(4));
<immutable digraph with 12 vertices, 24 edges>
gap> DigraphOrbitReps(D);
[ 1 ]
gap> D := DigraphMutableCopy(D);
<mutable digraph with 12 vertices, 24 edges>
gap> DigraphOrbitReps(D);
[ 1 ]
gap> D := DigraphFromDigraph6String("&IGO??S?'?_@?a?CK?O");
<immutable digraph with 10 vertices, 14 edges>
gap> DigraphOrbitReps(D);
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ]
gap> DigraphOrbitReps(DigraphMutableCopy(D));
[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ]
\end{verbatim}
7.2.13 DigraphSchreierVector

\[ \text{DigraphSchreierVector}(\text{digraph}) \]

**Returns:** An immutable list of integers.

DigraphSchreierVector returns the so-called Schreier vector of the action of the DigraphGroup (7.2.10) on the set of vertices of digraph. The Schreier vector is a list sch of integers with length DigraphNrVertices(digraph) where:

- \( \text{sch}[i] < 0 \): implies that \( i \) is an orbit representative and DigraphOrbitReps(digraph)[\(-\text{sch}[i]\)] = i.
- \( \text{sch}[i] > 0 \): implies that \( i \) / gens[\text{sch}[i]] is one step closer to the root (or representative) of the tree, where gens is the generators of DigraphGroup(digraph).

**Example**

\[
\begin{align*}
\text{gap> } & n := 4;; \\
\text{gap> } & \text{adj} := \text{function}(x, y) \text{ return } (((x - y) \text{ mod } n) = 1) \text{ or } (((x - y) \text{ mod } n) = n - 1); \\
\text{gap> } & \text{group} := \text{CyclicGroup(IsPermGroup, } n); \\
\text{gap> } & \text{Group([ (1,2,3,4) ])} \\
\text{gap> } & \text{D} := \text{Digraph(IsMutableDigraph, group, [1 .. n], \^, adj); } \\
\text{<mutable digraph with 4 vertices, 8 edges>} \\
\text{gap> } & \text{sch} := \text{DigraphSchreierVector(D)}; \\
\text{[ -1, 2, 2, 1 ]} \\
\text{gap> } & \text{D} := \text{CayleyDigraph(AlternatingGroup(4)); } \\
\text{<immutable digraph with 12 vertices, 24 edges>} \\
\text{gap> } & \text{sch} := \text{DigraphSchreierVector(D)}; \\
\text{[ -1, 2, 2, 1, 1, 2, 1, 2, 2, 1, 1, 1 ]} \\
\text{gap> } & \text{DigraphOrbitReps(D)}; \\
\text{[ 1 ]} \\
\text{gap> } & \text{gens} := \text{GeneratorsOfGroup(DigraphGroup(D)); } \\
\text{[ (1,7,5)(2,10,9)(3,4,11)(6,8,12), (1,3,2)(4,5,6)(7,9,8)(10,11,12) ]} \\
\text{gap> } & 10 / \text{gens[\text{sch}[10]]} \\
\text{2} \\
\text{gap> } & 7 / \text{gens[\text{sch}[7]]}; \\
\text{1} \\
\text{gap> } & 5 / \text{gens[\text{sch}[5]]}; \\
\text{7}
\end{align*}
\]

7.2.14 DigraphStabilizer

\[ \text{DigraphStabilizer}(\text{digraph, v}) \]

**Returns:** A permutation group.

DigraphStabilizer returns the stabilizer of the vertex \( v \) under of the action of the DigraphGroup (7.2.10) on the set of vertices of digraph.

**Example**

\[
\begin{align*}
\text{gap> } & \text{D} := \text{DigraphFromDigraph6String("&GYHPQgWTIIPW"); } \\
\text{<immutable digraph with 8 vertices, 24 edges>} \\
\text{gap> } & \text{DigraphStabilizer(D, 8); } \\
\text{Group(())}
\end{align*}
\]
Digraphs

```gap
gap> DigraphStabilizer(D, 2);
Group(())
gap> D := DigraphMutableCopy(D);
<mutable digraph with 8 vertices, 24 edges>
gap> DigraphStabilizer(D, 8);
Group(())
gap> DigraphStabilizer(D, 2);
Group(())
```

### 7.2.15 IsIsomorphicDigraph (for digraphs)

▷ IsIsomorphicDigraph(digraph1, digraph2)

(operations)

**Returns:** true or false.

This operation returns true if there exists an isomorphism from the digraph `digraph1` to the digraph `digraph2`. See `IsomorphismDigraphs (7.2.17)` for more information about isomorphisms of digraphs.

By default, an isomorphism is found using the canonical labellings of the digraphs obtained from bliss by Tommi Junttila and Petteri Kaski. If `NautyTracesInterface` is available, then `nauty` by Brendan McKay and Adolfo Piperno can be used instead; see `DigraphsUseBliss (7.2.1)`, and `DigraphsUseNauty (7.2.1)`.

```gap
gap> digraph1 := CycleDigraph(4);
<immutable cycle digraph with 4 vertices>
gap> digraph2 := CycleDigraph(5);
<immutable cycle digraph with 5 vertices>
gap> IsIsomorphicDigraph(digraph1, digraph2);
false
gap> digraph2 := DigraphReverse(digraph1);
<immutable digraph with 4 vertices, 4 edges>
gap> IsIsomorphicDigraph(digraph1, digraph2);
true
gap> digraph1 := Digraph([[3], [], []]);
<immutable digraph with 3 vertices, 1 edge>
gap> digraph2 := Digraph([[], [], [2]]);
<immutable digraph with 3 vertices, 1 edge>
gap> IsIsomorphicDigraph(digraph1, digraph2);
true
```

### 7.2.16 IsIsomorphicDigraph (for digraphs and homogeneous lists)

▷ IsIsomorphicDigraph(digraph1, digraph2, colours1, colours2)

(operations)

**Returns:** true or false.

This operation tests for isomorphism of coloured digraphs. A coloured digraph can be specified by its underlying digraph `digraph1` and its colouring `colours1`. Let \( n \) be the number of vertices of `digraph1`. The colouring `colours1` may have one of the following two forms:

- a list of \( n \) integers, where `colours[i]` is the colour of vertex \( i \), using the colours \([1 .. m]\) for some \( m \leq n \); or
• a list of non-empty disjoint lists whose union is DigraphVertices(digraph), such that 
colours[i] is the list of all vertices with colour i.

If digraph1 and digraph2 are digraphs without multiple edges, and colours1 and colours2 are 
colourings of digraph1 and digraph2, respectively, then this operation returns true if there exists 
an isomorphism between these two coloured digraphs. See IsomorphismDigraphs (7.2.18) for more 
information about isomorphisms of coloured digraphs.

By default, an isomorphism is found using the canonical labellings of the digraphs obtained 
from bliss by Tommi Junttila and Petteri Kaski. If NautyTracesInterface is available, then nauty 
by Brendan McKay and Adolfo Piperno can be used instead; see DigraphsUseBliss (7.2.1), and 
DigraphsUseNauty (7.2.1).
7.2.18 IsomorphismDigraphs (for digraphs and homogeneous lists)

▷ IsomorphismDigraphs(digraph1, digraph2, colours1, colours2) (operation)

Returns: A permutation, or fail.

This operation searches for an isomorphism between coloured digraphs. A coloured digraph can be specified by its underlying digraph `digraph1` and its colouring `colours1`. Let n be the number of vertices of `digraph1`. The colouring `colours1` may have one of the following two forms:

- a list of n integers, where `colours[i]` is the colour of vertex i, using the colours [1 .. m] for some m <= n; or

- a list of non-empty disjoint lists whose union is `DigraphVertices(digraph)`, such that `colours[i]` is the list of all vertices with colour i.

An isomorphism between coloured digraphs is an isomorphism between the underlying digraphs that preserves the colourings. See `IsomorphismDigraphs (7.2.17)` for more information about isomorphisms of digraphs. More precisely, let f be an isomorphism of digraphs from the digraph `digraph1` (with colouring `colours1`) to the digraph `digraph2` (with colouring `colours2`), and let p be the permutation of the vertices of `digraph1` that corresponds to f. Then f preserves the colourings of `digraph1` and `digraph2` – and hence is an isomorphism of coloured digraphs – if `colours1[i] = colours2[i ^ p]` for all vertices i in `digraph1`.

This operation returns such an isomorphism if one exists, else it returns `fail`.

By default, an isomorphism is found using the canonical labellings of the digraphs obtained from bliss by Tommi Junttila and Petteri Kaski. If `NautyTracesInterface` is available, then `nauty` by Brendan Mckay and Adolfo Piperno can be used instead; see `DigraphsUseBliss (7.2.1)`, and `DigraphsUseNauty (7.2.1).`
```
gap> p := IsomorphismDigraphs(digraph1, digraph2, colours1, colours2);
(1,4)(2,3)
gap> OnDigraphs(digraph1, p) = digraph2;
true
gap> List(DigraphVertices(digraph1), i -> colours1[i ^ p]) = colours2;
true
gap> colours1 := [1, 1, 2, 2];;
gap> colours2 := [2, 2, 1, 1];;
gap> p := IsomorphismDigraphs(digraph1, digraph2, colours1, colours2);
(1,4)(2,3)
gap> OnDigraphs(digraph1, p) = digraph2;
true
gap> List(DigraphVertices(digraph1), i -> colours1[i ^ p]) = colours2;
true
gap> IsomorphismDigraphs(digraph1, digraph2, [1, 1, 2, 2], [1, 1, 2, 2]);
fail
```

### 7.2.19 RepresentativeOutNeighbours

**RepresentativeOutNeighbours**(digraph)  

**Returns:** An immutable list of lists.

This function returns the list out of out-neighbours of each representative of the orbits of the action of DigraphGroup (7.2.10) on the vertex set of the digraph digraph.

More specifically, if reps is the list of orbit representatives, then a vertex j appears in out[i] each time there exists an edge with source reps[i] and range j in digraph.

If DigraphGroup (7.2.10) is trivial, then OutNeighbours (5.2.6) is returned.

**Example**

```
gap> D := Digraph([  
> [2, 1, 3, 4, 5], [3, 5], [2], [1, 2, 3, 5], [1, 2, 3, 4]]);
<immutable digraph with 5 vertices, 16 edges>
gap> DigraphGroup(D);
Group(())
gap> RepresentativeOutNeighbours(D);
[ [ 2, 1, 3, 4, 5 ], [ 3, 5 ], [ 2 ], [ 1, 2, 3, 5 ], [ 1, 2, 3, 4 ] ]
gap> D := Digraph(IsMutableDigraph, [  
> [2, 1, 3, 4, 5], [3, 5], [2], [1, 2, 3, 5], [1, 2, 3, 4]]);
<mutable digraph with 5 vertices, 16 edges>
gap> DigraphGroup(D);
Group(())
gap> RepresentativeOutNeighbours(D);
[ [ 2, 1, 3, 4, 5 ], [ 3, 5 ], [ 2 ], [ 1, 2, 3, 5 ], [ 1, 2, 3, 4 ] ]
gap> D := DigraphFromDigraph6String("&GYHPQgWTIIPW");
<immutable digraph with 8 vertices, 24 edges>
gap> G := DigraphGroup(D);;
gap> GeneratorsOfGroup(G);
[ [ 1, 2, 3, 4, 5, 6 ] ]
gap> Set(RepresentativeOutNeighbours(D), Set);
[ [ 2, 3, 5 ] ]```
7.2.20  IsDigraphIsomorphism (for digraphs and transformation or permutation)

▷ IsDigraphIsomorphism(src, ran, x)  
  (operation)
▷ IsDigraphIsomorphism(src, ran, x, col1, col2)  
  (operation)
▷ IsDigraphAutomorphism(digraph, x)  
  (operation)
▷ IsDigraphAutomorphism(digraph, x, col)  
  (operation)

**Returns:** true or false.

IsDigraphIsomorphism returns true if the permutation or transformation \( x \) is an isomorphism from the digraph \( src \) to the digraph \( ran \).

IsDigraphAutomorphism returns true if the permutation or transformation \( x \) is an automorphism of the digraph \( digraph \).

A permutation or transformation \( x \) is an *isomorphism* from a digraph \( src \) to a digraph \( ran \) if the following hold:

- \( x \) is a bijection from the vertices of \( src \) to those of \( ran \);
- \([u \sim x, v \sim x]\) is an edge of \( ran \) if and only if \([u, v]\) is an edge of \( src \); and
- \( x \) fixes every \( i \) which is not a vertex of \( src \).

See also AutomorphismGroup (7.2.2).

If \( col1 \) and \( col2 \), or \( col \), are given, then they must represent vertex colourings; see AutomorphismGroup (7.2.5) for details of the permissible values for these arguments. The homomorphism must then also have the property:

- \( col1[i] = col2[i \sim x] \) for all vertices \( i \) of \( src \), for IsDigraphIsomorphism.
- \( col[i] = col[i \sim x] \) for all vertices \( i \) of \( digraph \), for IsDigraphAutomorphism.

For some digraphs, it can be faster to use IsDigraphAutomorphism than to test membership in the automorphism group of \( digraph \).

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>gap&gt; src := Digraph([[1], [1, 2], [1, 3]]); &lt;- immutable digraph with 3 vertices, 5 edges&gt;</td>
</tr>
<tr>
<td>false</td>
</tr>
<tr>
<td>gap&gt; IsDigraphAutomorphism(src, (1, 2, 3)); &lt;- true</td>
</tr>
<tr>
<td>gap&gt; IsDigraphAutomorphism(src, (2, 3)); &lt;- true</td>
</tr>
<tr>
<td>gap&gt; IsDigraphAutomorphism(src, (2, 3), [2, 1, 1]); &lt;- true</td>
</tr>
<tr>
<td>gap&gt; IsDigraphAutomorphism(src, (2, 3), [2, 2, 1]); &lt;- false</td>
</tr>
<tr>
<td>gap&gt; IsDigraphAutomorphism(src, (2, 3)(4, 5)); &lt;- false</td>
</tr>
<tr>
<td>gap&gt; IsDigraphAutomorphism(src, (1, 4)); &lt;- false</td>
</tr>
<tr>
<td>gap&gt; IsDigraphAutomorphism(src, ()); &lt;- true</td>
</tr>
<tr>
<td>gap&gt; ran := Digraph([[2, 1], [2], [2, 3]]); &lt;- immutable digraph with 3 vertices, 5 edges&gt;</td>
</tr>
<tr>
<td>gap&gt; IsDigraphIsomorphism(src, ran, (1, 2)); &lt;- true</td>
</tr>
</tbody>
</table>
7.2.21  IsDigraphColouring

▷ IsDigraphColouring(digraph, list)  
▷ IsDigraphColouring(digraph, t)  

Returns: true or false.

The operation IsDigraphColouring verifies whether or not the list list describes a proper colouring of the digraph digraph.

A list list describes a proper colouring of a digraph digraph if list consists of positive integers, the length of list equals the number of vertices in digraph, and for any vertices u, v of digraph if u and v are adjacent, then list[u] >> list[v].

A transformation t describes a proper colouring of a digraph digraph, if ImageListOfTransformation(t, DigraphNrVertices(digraph)) is a proper colouring of digraph.

See also IsDigraphHomomorphism (7.3.10).

Example

gap> D := JohnsonDigraph(5, 3);  
<immutable symmetric digraph with 10 vertices, 60 edges> 
gap> IsDigraphColouring(D, [1, 2, 3, 3, 2, 1, 4, 5, 6, 7]);  
true 
gap> IsDigraphColouring(D, [1, 2, 3, 3, 2, 1, 2, 5, 6, 7]);  
false 
gap> IsDigraphColouring(D, [1, 2, 3, 3, 2, 1, 2, 5, 6, -1]);  
false 
gap> IsDigraphColouring(D, [1, 2, 3]);  
false 
gap> IsDigraphColouring(D, IdentityTransformation);  
true

7.2.22  MaximalCommonSubdigraph

▷ MaximalCommonSubdigraph(D1, D2)  

Returns: A list containing a digraph and two transformations.

If D1 and D2 are digraphs without multiple edges, then MaximalCommonSubdigraph returns a maximal common subdigraph M of D1 and D2 with the maximum number of vertices. So M is a digraph which embeds into both D1 and D2 and has the largest number of vertices among such digraphs. It returns a list [M, t1, t2] where M is the maximal common subdigraph and t1, t2 are transformations embedding M into D1 and D2 respectively.
Digraphs

Example

```
gap> MaximalCommonSubdigraph(PetersenGraph(), CompleteDigraph(10));
[ <immutable digraph with 2 vertices, 2 edges>,
  IdentityTransformation, IdentityTransformation ]
gap> MaximalCommonSubdigraph(PetersenGraph(),
  DigraphSymmetricClosure(CycleDigraph(5)));
[ <immutable digraph with 5 vertices, 10 edges>,
  IdentityTransformation, IdentityTransformation ]
gap> MaximalCommonSubdigraph(NullDigraph(0), CompleteDigraph(10));
[ <immutable empty digraph with 0 vertices>, IdentityTransformation,
  IdentityTransformation ]
```

7.2.23 MinimalCommonSuperdigraph

▷ MinimalCommonSuperdigraph(D1, D2)

(operation)

Returns: A list containing a digraph and two transformations.

If D1 and D2 are digraphs without multiple edges, then MinimalCommonSuperdigraph returns a minimal common superdigraph M of D1 and D2 with the minimum number of vertices. So M is a digraph into which both D1 and D2 embed and has the smallest number of vertices among such digraphs. It returns a list [M, t1, t2] where M is the minimal common superdigraph and t1, t2 are transformations embedding D1 and D2 respectively into M.

```
gap> MinimalCommonSuperdigraph(PetersenGraph(), CompleteDigraph(10));
[ <immutable digraph with 18 vertices, 118 edges>,
  Transformation([ 1, 2, 11, 12, 13, 14, 15, 16, 17, 18, 11, 12, 13,
  14, 15, 16, 17, 18 ] ) ]
gap> MinimalCommonSuperdigraph(PetersenGraph(),
  DigraphSymmetricClosure(CycleDigraph(5)));
[ <immutable digraph with 10 vertices, 30 edges>,
  IdentityTransformation, IdentityTransformation ]
gap> MinimalCommonSuperdigraph(NullDigraph(0), CompleteDigraph(10));
[ <immutable empty digraph with 0 vertices>, IdentityTransformation,
  IdentityTransformation ]
```

7.3 Homomorphisms of digraphs

The following methods exist to find homomorphisms between digraphs. If an argument to one of these methods is a digraph with multiple edges, then the multiplicity of edges will be ignored in order to perform the calculation; the digraph will be treated as if it has no multiple edges.

7.3.1 HomomorphismDigraphsFinder

▷ HomomorphismDigraphsFinder(D1, D2, hook, user_param, max_results, hint,
  injective, image, partial_map, colors1, colors2[, order, aut_grp])

Returns: The argument user_param.

This function finds homomorphisms from the digraph D1 to the digraph D2 subject to the conditions imposed by the other arguments as described below.
If \( f \) and \( g \) are homomorphisms found by \texttt{HomomorphismDigraphsFinder}, then \( f \) cannot be obtained from \( g \) by right multiplying by an automorphism of \( D_2 \) in \texttt{aut_grp}.

\texttt{hook}

This argument should be a function or \texttt{fail}.

If \texttt{hook} is a function, then it must have two arguments \texttt{user_param} (see below) and a transformation \( t \). The function \texttt{hook(user_param, t)} is called every time a new homomorphism \( t \) is found by \texttt{HomomorphismDigraphsFinder}. If the function returns \texttt{true}, then \texttt{HomomorphismDigraphsFinder} stops and does not find any further homomorphisms. This feature might be useful if you are searching for a homomorphism that satisfies some condition that you cannot specify via the other arguments to \texttt{HomomorphismDigraphsFinder}.

If \texttt{hook} is \texttt{fail}, then a default function is used which simply adds every new homomorphism found by \texttt{HomomorphismDigraphsFinder} to \texttt{user_param}, which must be a mutable list in this case.

\texttt{user_param}

If \texttt{hook} is a function, then \texttt{user_param} can be any \texttt{GAP} object. The object \texttt{user_param} is used as the first argument of the function \texttt{hook}. For example, \texttt{user_param} might be a transformation semigroup, and \texttt{hook(user_param, t)} might set \texttt{user_param} to be the closure of \texttt{user_param} and \( t \).

If the value of \texttt{hook} is \texttt{fail}, then the value of \texttt{user_param} must be a mutable list.

\texttt{max_results}

This argument should be a positive integer or \texttt{infinity}. \texttt{HomomorphismDigraphsFinder} will return after it has found \texttt{max_results} homomorphisms or the search is complete, whichever happens first.

\texttt{hint}

This argument should be a positive integer or \texttt{fail}.

If \texttt{hint} is a positive integer, then only homomorphisms of rank \texttt{hint} are found.

If \texttt{hint} is \texttt{fail}, then no restriction is put on the rank of homomorphisms found.

\texttt{injective}

This argument should be 0, 1, or 2. If it is 2, then only embeddings are found, if it is 1, then only injective homomorphisms are found, and if it is 0 there are no restrictions imposed by this argument.

For backwards compatibility, \texttt{injective} can also be \texttt{false} or \texttt{true} which correspond to the values 0 and 1 described in the previous paragraph, respectively.

\texttt{image}

This argument should be a subset of the vertices of the graph \( D_2 \). \texttt{HomomorphismDigraphsFinder} only finds homomorphisms from \( D_1 \) to the subgraph of \( D_2 \) induced by the vertices \texttt{image}.

\texttt{partial_map}

This argument should be a partial map from \( D_1 \) to \( D_2 \), that is, a (not necessarily dense) list of vertices of the digraph \( D_2 \) of length no greater than the number vertices in the digraph \( D_1 \). \texttt{HomomorphismDigraphsFinder} only finds homomorphisms extending \texttt{partial_map} (if any).
colors1
This should be a list representing possible colours of vertices in the digraph $D_1$; see AutomorphismGroup (7.2.5) for details of the permissible values for this argument.

colors2
This should be a list representing possible colours of vertices in the digraph $D_2$; see AutomorphismGroup (7.2.5) for details of the permissible values for this argument.

order
The optional argument order specifies the order the vertices in $D_1$ appear in the search for homomorphisms. The value of this parameter can have a large impact on the runtime of the function. It seems in many cases to be a good idea for this to be the DigraphWelshPowellOrder (7.3.16), i.e. vertices ordered from highest to lowest degree. The optional argument aut_grp should be a subgroup of the automorphism group of $D_2$. This function returns unique representatives of the homomorphisms found up to right multiplication by aut_grp. If this argument is not specific, it defaults to the full automorphism group of $D_2$, which may be costly to calculate.

Example

```gap
d := ChainDigraph(10);
<immutable chain digraph with 10 vertices>
d := DigraphSymmetricClosure(d);
<immutable symmetric digraph with 10 vertices, 18 edges>
HomomorphismDigraphsFinder(d, d, fail, [], infinity, 2, 0,
> [3, 4], [], fail, fail);
[ Transformation([3, 4, 3, 4, 3, 4, 3, 4, 3, 4]),
  Transformation([4, 3, 4, 3, 4, 3, 4, 3, 4, 3]) ]
d := CompleteDigraph(6);
HomomorphismDigraphsFinder(d, d, fail, [], 1, fail, 0,
> [1 .. 6], [1, 2, 1], fail, fail);
[ Transformation([1, 2, 1, 3, 4, 5, 6, 1, 2, 1]) ]
f := function(user_param, t)
> Add(user_param, t * user_param[1]);
> end;

HomomorphismDigraphsFinder(d, d, f, Transformation([2, 2]),
> 3, fail, 0, [1 .. 6], [1, 2, 1], fail, fail);
[ Transformation([2, 2]),
  Transformation([2, 2, 2, 3, 4, 5, 6, 2, 2, 2]),
  Transformation([2, 2, 2, 3, 4, 5, 6, 2, 2, 3]),
  Transformation([2, 2, 2, 3, 4, 5, 6, 2, 2, 4]) ]
HomomorphismDigraphsFinder(NullDigraph(2), NullDigraph(3), fail,
> [], infinity, fail, 1, [1, 2, 3], fail, fail, fail,
> Group((1, 2)));
[ IdentityTransformation, Transformation([1, 3, 3]),
  Transformation([2, 1]), Transformation([2, 3, 3]),
  Transformation([3, 1, 3]), Transformation([3, 2, 3]) ]
HomomorphismDigraphsFinder(NullDigraph(2), NullDigraph(3), fail,
> [], infinity, fail, 1, [1, 2, 3], fail, fail, fail, fail,
> Group((1, 2)));
[ IdentityTransformation, Transformation([1, 3, 3]),
  Transformation([3, 1, 3]) ]
```
7.3.2 DigraphHomomorphism

\[\text{DigraphHomomorphism(digraph1, digraph2)}\]

Returns: A transformation, or fail.

A homomorphism from \(\text{digraph1}\) to \(\text{digraph2}\) is a mapping from the vertex set of \(\text{digraph1}\) to a subset of the vertices of \(\text{digraph2}\), such that every pair of vertices \([i,j]\) which has an edge \(i\rightarrow j\) is mapped to a pair of vertices \([a,b]\) which has an edge \(a\rightarrow b\). Note that non-adjacent vertices can still be mapped to adjacent vertices.

\text{DigraphHomomorphism} \text{returns a single homomorphism between} \text{digraph1} \text{and} \text{digraph2} \text{if it exists, otherwise it returns fail.}

Example

\begin{verbatim}
gap> gr1 := ChainDigraph(3);;
gap> gr2 := Digraph([[3, 5], [2], [3, 1], [], [4]]);
<immutable digraph with 5 vertices, 6 edges>
gap> DigraphHomomorphism(gr1, gr1);
IdentityTransformation
gap> map := DigraphHomomorphism(gr1, gr2);
Transformation([3, 1, 5, 4, 5])
gap> IsDigraphHomomorphism(gr1, gr2, map);
true
\end{verbatim}

7.3.3 HomomorphismsDigraphs

\[\text{HomomorphismsDigraphs(digraph1, digraph2)}\]

\[\text{HomomorphismsDigraphsRepresentatives(digraph1, digraph2)}\]

Returns: A list of transformations.

\text{HomomorphismsDigraphsRepresentatives} \text{finds every} \text{DigraphHomomorphism} (7.3.2) \text{between} \text{digraph1} \text{and} \text{digraph2}, \text{up to right multiplication by an element of the AutomorphismGroup} (7.2.2) \text{of} \text{digraph2}. \text{In other words, every homomorphism} \text{f} \text{between} \text{digraph1} \text{and} \text{digraph2} \text{can be written as the composition} \text{f = g * x}, \text{where} \text{g} \text{is one of the} \text{HomomorphismsDigraphsRepresentatives} \text{and} \text{x} \text{is an automorphism of} \text{digraph2}.

\text{HomomorphismsDigraphs} \text{returns all homomorphisms between} \text{digraph1} \text{and} \text{digraph2}.

Example

\begin{verbatim}
gap> gr1 := ChainDigraph(3);;
gap> gr2 := Digraph([[3, 5], [2], [3, 1], [], [4]]);
<immutable digraph with 5 vertices, 6 edges>
gap> HomomorphismsDigraphs(gr1, gr2);
[ Transformation([1, 3, 1]), Transformation([1, 3, 3]), Transformation([1, 5, 4, 4, 5]), Transformation([2, 2, 2]), Transformation([3, 1, 3]), Transformation([3, 1, 5, 4, 5]), Transformation([3, 3, 1]), Transformation([3, 3, 3]) ]
gap> HomomorphismsDigraphsRepresentatives(gr1, CompleteDigraph(3));
[ Transformation([2, 1]), Transformation([2, 1, 2]) ]
\end{verbatim}

7.3.4 DigraphMonomorphism

\[\text{DigraphMonomorphism(digraph1, digraph2)}\]

Returns: A transformation, or fail.

\text{DigraphMonomorphism} \text{returns a single} \text{injective} \text{DigraphHomomorphism} (7.3.2) \text{between} \text{digraph1} \text{and} \text{digraph2} \text{if one exists, otherwise it returns fail.}
7.3.5 MonomorphismsDigraphs

MonomorphismsDigraphs(digraph1, digraph2) (operation)

MonomorphismsDigraphsRepresentatives(digraph1, digraph2) (operation)

Returns: A list of transformations.

These operations behave the same as HomomorphismsDigraphs (7.3.3) and HomomorphismsDigraphsRepresentatives (7.3.3), except they only return injective homomorphisms.

Example

```gap
gap> gr1 := ChainDigraph(3);
<immutable digraph with 3 vertices, 3 edges>
gap> gr2 := Digraph([[3, 5], [2], [3, 1], [], [4]]);
<immutable digraph with 5 vertices, 6 edges>
gap> MonomorphismsDigraphs(gr1, gr2);
[ Transformation([1, 5, 4, 5]),
  Transformation([3, 1, 5, 4, 5]) ]
gap> MonomorphismsDigraphsRepresentatives(gr1, CompleteDigraph(3));
[ Transformation([2, 1]) ]
```

7.3.6 DigraphEpimorphism

DigraphEpimorphism(digraph1, digraph2) (operation)

Returns: A transformation, or fail.

DigraphEpimorphism returns a single surjective DigraphHomomorphism (7.3.2) between digraph1 and digraph2 if one exists, otherwise it returns fail.

Example

```gap
gap> gr1 := DigraphReverse(ChainDigraph(4));
<mutable digraph with 4 vertices, 3 edges>
gap> gr2 := DigraphRemoveEdge(CompleteDigraph(3), [1, 2]);
<mutable digraph with 3 vertices, 5 edges>
gap> DigraphEpimorphism(gr2, gr1);
fail
gap> DigraphEpimorphism(gr1, gr2);
Transformation([1, 2, 3])
```

7.3.7 EpimorphismsDigraphs

EpimorphismsDigraphs(digraph1, digraph2) (operation)

EpimorphismsDigraphsRepresentatives(digraph1, digraph2) (operation)

Returns: A list of transformations.
These operations behave the same as HomomorphismsDigraphs (7.3.3) and HomomorphismsDigraphsRepresentatives (7.3.3), except they only return surjective homomorphisms.

Example

```gap
> gr1 := DigraphReverse(ChainDigraph(4));  
<immutable digraph with 4 vertices, 3 edges>
> gr2 := DigraphSymmetricClosure(CycleDigraph(3));  
<immutable symmetric digraph with 3 vertices, 6 edges>
> EpimorphismsDigraphsRepresentatives(gr1, gr2);  
[ Transformation( [ 3, 1, 2, 1 ] ), Transformation( [ 3, 1, 2, 3 ] ),  
  Transformation( [ 2, 1, 2, 3 ] ) ]
> EpimorphismsDigraphs(gr1, gr2);  
[ Transformation( [ 3, 1, 2, 1 ] ), Transformation( [ 3, 1, 2, 3 ] ),  
  Transformation( [ 2, 1, 2, 3 ] ) ]
```

7.3.8 DigraphEmbedding

- **DigraphEmbedding(digraph1, digraph2)**
  - **Returns**: A transformation, or fail.

  An embedding of a digraph `digraph1` into another digraph `digraph2` is a DigraphMonomorphism (7.3.4) from `digraph1` to `digraph2` which has the additional property that a pair of vertices `[i, j]` which have no edge `i -> j` in `digraph1` are mapped to a pair of vertices `[a, b]` which have no edge `a -> b` in `digraph2`.

  In other words, an embedding `t` is an isomorphism from `digraph1` to the InducedSubdigraph (3.3.3) of `digraph2` on the image of `t`.

  `DigraphEmbedding` returns a single embedding if one exists, otherwise it returns fail.

Example

```gap
> gr := ChainDigraph(3);  
<immutable chain digraph with 3 vertices>
> DigraphEmbedding(gr, CompleteDigraph(4));  
fail
> DigraphEmbedding(gr, Digraph([[3], [1, 4], [1], [3]]));  
Transformation( [ 2, 4, 3, 4 ] )
```

7.3.9 EmbeddingsDigraphs

- **EmbeddingsDigraphs(D1, D2)**
  - (operation)

  These operations behave the same as HomomorphismsDigraphs (7.3.3) and HomomorphismsDigraphsRepresentatives (7.3.3), except they only return embeddings of `D1` into `D2`.

  See also IsDigraphEmbedding (7.3.11).
Example

```gap
gap> D1 := NullDigraph(2);
<immutable empty digraph with 2 vertices>
gap> D2 := CycleDigraph(5);
<immutable cycle digraph with 5 vertices>
gap> EmbeddingsDigraphsRepresentatives(D1, D2);
[ Transformation( [ 1, 3, 3 ] ), Transformation( [ 1, 4, 3, 4 ] ) ]
gap> EmbeddingsDigraphs(D1, D2);
[ Transformation( [ 1, 3, 3 ] ), Transformation( [ 1, 4, 3, 4 ] ),
  Transformation( [ 2, 4, 4, 5, 1 ] ),
  Transformation( [ 2, 5, 4, 5, 1 ] ),
  Transformation( [ 3, 1, 5, 1, 2 ] ),
  Transformation( [ 3, 5, 5, 1, 2 ] ),
  Transformation( [ 4, 1, 1, 2, 3 ] ),
  Transformation( [ 4, 2, 1, 2, 3 ] ),
  Transformation( [ 5, 2, 2, 3, 4 ] ),
  Transformation( [ 5, 3, 2, 3, 4 ] ) ]
```

7.3.10  IsDigraphHomomorphism (for digraphs and a permutation or transformation)

▷ IsDigraphHomomorphism(src, ran, x)  (operation)
▷ IsDigraphHomomorphism(src, ran, x, col1, col2)  (operation)
▷ IsDigraphEpimorphism(src, ran, x)  (operation)
▷ IsDigraphEpimorphism(src, ran, x, col1, col2)  (operation)
▷ IsDigraphMonomorphism(src, ran, x)  (operation)
▷ IsDigraphMonomorphism(src, ran, x, col1, col2)  (operation)
▷ IsDigraphEndomorphism(digraph, x)  (operation)
▷ IsDigraphEndomorphism(digraph, x, col)  (operation)

Returns: true or false.

IsDigraphHomomorphism returns true if the permutation or transformation \( x \) is a homomorphism from the digraph \( src \) to the digraph \( ran \).

IsDigraphEpimorphism returns true if the permutation or transformation \( x \) is a surjective homomorphism from the digraph \( src \) to the digraph \( ran \).

IsDigraphMonomorphism returns true if the permutation or transformation \( x \) is an injective homomorphism from the digraph \( src \) to the digraph \( ran \).

IsDigraphEndomorphism returns true if the permutation or transformation \( x \) is an endomorphism of the digraph \( digraph \).

A permutation or transformation \( x \) is a homomorphism from a digraph \( src \) to a digraph \( ran \) if the following hold:

- \([u \sim x, v \sim x]\) is an edge of \( ran \) whenever \([u, v]\) is an edge of \( src \); and
- \( x \) fixes every \( i \) which is not a vertex of \( src \).

See also GeneratorsOfEndomorphismMonoid (7.3.13).

If \( col1 \) and \( col2 \), or \( col \), are given, then they must represent vertex colourings; see AutomorphismGroup (7.2.5) for details of the permissible values for these argument. The homomorphism must then also have the property:
• \( \text{col}[i] = \text{col}[i \circ x] \) for all vertices \( i \) of digraph, in the case of \( \text{IsDigraphEndomorphism} \).

• \( \text{col1}[i] = \text{col2}[i \circ x] \) for all vertices \( i \) of \( \text{src} \), in the cases of the other operations.

See also \( \text{DigraphsRespectsColouring} \) (7.3.12).

Example

\[
\begin{align*}
gap & \text{src := Digraph([[1], [1, 2], [1, 3]]);} \\
gap & \text{<immutable digraph with 3 vertices, 5 edges>} \\
gap & \text{ran := Digraph([[1], [1, 2]]);} \\
gap & \text{<immutable digraph with 2 vertices, 3 edges>} \\
gap & \text{IsDigraphHomomorphism(src, ran, Transformation([1, 2, 2]));} \\
gap & \text{true} \\
gap & \text{IsDigraphHomomorphism(src, ran, Transformation([2, 1, 2]));} \\
gap & \text{false} \\
gap & \text{IsDigraphHomomorphism(src, ran, Transformation([3, 3, 3]));} \\
gap & \text{false} \\
gap & \text{IsDigraphHomomorphism(src, src, Transformation([3, 3, 3]));} \\
gap & \text{true} \\
gap & \text{IsDigraphHomomorphism(src, ran, Transformation([1, 2, 2]),} \\
gap & \quad [1, 2, 2], [1, 2]); \\
gap & \quad \text{true} \\
gap & \text{IsDigraphHomomorphism(src, ran, Transformation([1, 2, 2]),} \\
gap & \quad [2, 1, 1], [1, 2]); \\
gap & \quad \text{false} \\
gap & \text{IsDigraphEndomorphism(src, Transformation([3, 3, 3]));} \\
gap & \text{true} \\
gap & \text{IsDigraphEndomorphism(src, Transformation([3, 3, 3]), [1, 1, 1]);} \\
gap & \text{true} \\
gap & \text{IsDigraphEndomorphism(src, Transformation([3, 3, 3]), [1, 1, 2]);} \\
gap & \text{false} \\
gap & \text{IsDigraphEpimorphism(src, ran, Transformation([3, 3, 3]));} \\
gap & \text{false} \\
gap & \text{IsDigraphMonomorphism(src, ran, Transformation([1, 2, 2]));} \\
gap & \text{false} \\
gap & \text{IsDigraphEpimorphism(src, ran, Transformation([1, 2, 2]));} \\
gap & \text{true} \\
gap & \text{IsDigraphMonomorphism(ran, src, ());} \\
gap & \text{true}
\end{align*}
\]

7.3.11 \( \text{IsDigraphEmbedding} \) (for digraphs and a permutation or transformation)

\( \text{IsDigraphEmbedding} \) returns \text{true} if the permutation or transformation \( x \) is a embedding of the digraph \( \text{src} \) into the digraph \( \text{ran} \), while respecting the colourings \( \text{col1} \) and \( \text{col2} \) if given.

A permutation or transformation \( x \) is an \text{embedding} of a digraph \( \text{src} \) into a digraph \( \text{ran} \) if \( x \) is a monomorphism from \( \text{src} \) to \( \text{ran} \) and the inverse of \( x \) is a monomorphism from the subdigraph of \( \text{ran} \) induced by the image of \( x \) to \( \text{src} \). See also \( \text{IsDigraphHomomorphism} \) (7.3.10).
Example

```gap
gap> src := Digraph([[1], [1, 2]]);
<immutable digraph with 2 vertices, 3 edges>
gap> ran := Digraph([[1], [1, 2], [1, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> IsDigraphMonomorphism(src, ran, ());
true
gap> IsDigraphEmbedding(src, ran, ());
true
gap> IsDigraphEmbedding(src, ran, (), [2, 1], [2, 1, 1]);
false
gap> ran := Digraph([[1, 2], [1, 2], [1, 3]]);
<immutable digraph with 3 vertices, 6 edges>
gap> IsDigraphMonomorphism(src, ran, IdentityTransformation);
true
gap> IsDigraphEmbedding(src, ran, IdentityTransformation);
false
```

7.3.12 DigraphsRespectsColouring

▷ DigraphsRespectsColouring(src, ran, x, col1, col2)  

(operation)

Returns: true or false.

The operation DigraphsRespectsColouring verifies whether or not the permutation or transformation \( x \) respects the vertex colourings \( \text{col1} \) and \( \text{col2} \) of the digraphs \( \text{src} \) and \( \text{range} \). That is, DigraphsRespectsColouring returns true if and only if for all vertices \( i \) of \( \text{src} \), \( \text{col1}[i] = \text{col2}[i ^ x] \).

Example

```gap
gap> src := Digraph([[1], [1, 2]]);
<immutable digraph with 2 vertices, 3 edges>
gap> ran := Digraph([[1], [1, 2], [1, 3]]);
<immutable digraph with 3 vertices, 5 edges>
gap> DigraphsRespectsColouring(src, ran, (1, 2), [2, 1], [1, 2, 1]);
true
gap> DigraphsRespectsColouring(src, ran, (1, 2), [2, 1], [2, 1, 1]);
false
```

7.3.13 GeneratorsOfEndomorphismMonoid

▷ GeneratorsOfEndomorphismMonoid(digraph[, colors][, limit])  

(function)

▷ GeneratorsOfEndomorphismMonoidAttr(digraph)  

(attribute)

Returns: A list of transformations.

An endomorphism of \( \text{digraph} \) is a homomorphism DigraphHomomorphism (7.3.2) from \( \text{digraph} \) back to itself. GeneratorsOfEndomorphismMonoid, called with a single argument, returns a generating set for the monoid of all endomorphisms of \( \text{digraph} \). If \( \text{digraph} \) belongs to IsImmutableDigraph (3.1.3), then the value of GeneratorsOfEndomorphismMonoid will not be recomputed on future calls.
If the \texttt{colors} argument is specified, then \texttt{GeneratorsOfEndomorphismMonoid} will return a generating set for the monoid of endomorphisms which respect the given colouring. The colouring \texttt{colors} can be in one of two forms:

- A list of positive integers of size the number of vertices of \texttt{digraph}, where \texttt{colors}[i] is the colour of vertex \texttt{i}.

- A list of lists, such that \texttt{colors}[i] is a list of all vertices with colour \texttt{i}.

If the \texttt{limit} argument is specified, then it will return only the first \texttt{limit} homomorphisms, where \texttt{limit} must be a positive integer or infinity.

\begin{verbatim}
Example

gap> gr := Digraph(List([1 .. 3], x -> [1 .. 3]));;
gap> GeneratorsOfEndomorphismMonoid(gr);
[ Transformation( [ 1, 3, 2 ] ), Transformation( [ 2, 1 ] ),
  IdentityTransformation, Transformation( [ 1, 2, 1 ] ),
  Transformation( [ 1, 2, 2 ] ), Transformation( [ 1, 1, 2 ] ),
  Transformation( [ 1, 1, 1 ] ) ]

gap> GeneratorsOfEndomorphismMonoid(gr, 3);
[ Transformation( [ 1, 3, 2 ] ), Transformation( [ 2, 1 ] ),
  IdentityTransformation ]

gap> gr := CompleteDigraph(3);

gap> GeneratorsOfEndomorphismMonoid(gr);
[ Transformation( [ 2, 3, 1 ] ), Transformation( [ 2, 1 ] ),
  IdentityTransformation ]

gap> GeneratorsOfEndomorphismMonoid(gr, [1, 2, 2]);
[ Transformation( [ 1, 3, 2 ] ), IdentityTransformation ]

gap> GeneratorsOfEndomorphismMonoid(gr, [[1], [2, 3]]);
[ Transformation( [ 1, 3, 2 ] ), IdentityTransformation ]
\end{verbatim}

7.3.14 \textbf{DigraphColouring (for a digraph and a number of colours)}

\begin{verbatim}
Example

gap> DigraphColouring(CompleteDigraph(5), 4);
fail

gap> DigraphColouring(ChainDigraph(10), 1);
fail
\end{verbatim}

\begin{verbatim}

digraphColouring(digraph, n)

Returns: A transformation, or \texttt{fail}.

A proper colouring of a digraph is a labelling of its vertices in such a way that adjacent vertices have different labels. A proper \texttt{n}-colouring is a proper colouring that uses exactly \texttt{n} colours. Equivalently, a proper \texttt{n}-colouring of a digraph can be defined to be a \texttt{DigraphEpimorphism} (7.3.6) from a digraph onto the complete digraph (with \texttt{n} vertices); see \texttt{CompleteDigraph} (3.5.11). Note that a digraph with loops (\texttt{DigraphHasLoops} (6.2.1)) does not have a proper \texttt{n}-colouring for any value \texttt{n}.

If \texttt{digraph} is a digraph and \texttt{n} is a non-negative integer, then \texttt{DigraphColouring(digraph, n)} returns an epimorphism from \texttt{digraph} onto the complete digraph with \texttt{n} vertices if one exists, else it returns \texttt{fail}.

See also \texttt{DigraphGreedyColouring} (7.3.15) and \texttt{IsBipartiteDigraph} (6.2.3).

\end{verbatim}
Digraphs

7.3.15 DigraphGreedyColouring (for a digraph and vertex order)

▷ DigraphGreedyColouring(digraph, order) (operation)
▷ DigraphGreedyColouring(digraph, func) (operation)
▷ DigraphGreedyColouring(digraph) (attribute)

Returns: A transformation, or fail.

A proper colouring of a digraph is a labelling of its vertices in such a way that adjacent vertices have different labels. Note that a digraph with loops (DigraphHasLoops (6.2.1)) does not have any proper colouring.

If digraph is a digraph and order is a dense list consisting of all of the vertices of digraph (in any order), then DigraphGreedyColouring uses a greedy algorithm with the specified order to obtain some proper colouring of digraph, which may not use the minimal number of colours.

If digraph is a digraph and func is a function whose argument is a digraph, and that returns a dense list order, then DigraphGreedyColouring(digraph, func) returns DigraphGreedyColouring(digraph, func(digraph)).

If the optional second argument (either a list or a function), is not specified, then DigraphWelshPowellOrder (7.3.16) is used by default.

See also DigraphColouring (7.3.14).

Example

```gap
gap> DigraphGreedyColouring(ChainDigraph(10));
Transformation([2, 1, 2, 1, 2, 1, 2, 1, 2, 1])
gap> DigraphGreedyColouring(ChainDigraph(10), [1 .. 10]);
Transformation([1, 2, 1, 2, 1, 2, 1, 2, 1, 2])
```

7.3.16 DigraphWelshPowellOrder

▷ DigraphWelshPowellOrder(digraph) (attribute)

Returns: A list of the vertices.

DigraphWelshPowellOrder returns a list of all of the vertices of the digraph digraph ordered according to the sum of the number of out- and in-neighbours, from highest to lowest.

Example

```gap
gap> DigraphWelshPowellOrder(Digraph([[4], [9], [9], []],
> [4, 6, 9], [1], [], []),
> [4, 5], [4, 5]));
[5, 9, 4, 1, 6, 10, 2, 3, 7, 8]
```

7.3.17 ChromaticNumber

▷ ChromaticNumber(digraph) (attribute)

Returns: A non-negative integer.
A proper colouring of a digraph is a labelling of its vertices in such a way that adjacent vertices have different labels. Equivalently, a proper digraph colouring can be defined to be a DigraphEpimorphism (7.3.6) from a digraph onto a complete digraph.

If digraph is a digraph without loops (see DigraphHasLoops (6.2.1), then ChromaticNumber returns the least non-negative integer n such that there is a proper colouring of digraph with n colours. In other words, for a digraph with at least one vertex, ChromaticNumber returns the least number n such that DigraphColouring(digraph, n) does not return fail. See DigraphColouring (7.3.14).

It is possible to select the algorithm to compute the chromatic number via the use of value options. The permitted algorithms and values to pass as options are:

- lawler - Lawler’s Algorithm [Law76]
- byskov - Byskov’s Algorithm [Bys02]

Example

```gap
gap> ChromaticNumber(NullDigraph(10));
1
gap> ChromaticNumber(CompleteDigraph(10));
10
gap> ChromaticNumber(CompleteBipartiteDigraph(5, 5));
2
gap> ChromaticNumber(Digraph([[], [3], [5], [2, 3], [4]]));
3
gap> ChromaticNumber(NullDigraph(0));
0
gap> D := PetersenGraph(IsMutableDigraph);
<mutable digraph with 10 vertices, 30 edges>
gap> ChromaticNumber(D);
3
gap> ChromaticNumber(CompleteDigraph(10) : lawler);
10
gap> ChromaticNumber(CompleteDigraph(10) : byskov);
10
```

7.3.18 DigraphCore

\[ \text{DigraphCore}(D) \]

**Returns:** A list of positive integers.

If D is a digraph, then DigraphCore returns a list of vertices corresponding to the core of D. In particular, the subdigraph of D induced by this list is isomorphic to the core of D.

The core of a digraph D is the minimal subdigraph C of D which is a homomorphic image of D. The core of a digraph is unique up to isomorphism.

Example

```gap
gap> D := DigraphSymmetricClosure(CycleDigraph(8));
<immutable symmetric digraph with 8 vertices, 16 edges>
gap> DigraphCore(D);
[ 1, 2 ]
gap> D := PetersenGraph();
<immutable digraph with 10 vertices, 30 edges>
```

```gap
```
Digraphs

[ 1 .. 10 ]

gap> D := Digraph(IsMutableDigraph, [[3], [3], [4], [5], [2]]);
<mutable digraph with 5 vertices, 5 edges>
gap> DigraphCore(D);
[ 2, 3, 4, 5 ]

7.3.19 LatticeDigraphEmbedding

▷ LatticeDigraphEmbedding(L1, L2) (operation)

Returns: A transformation, or fail.

If L1 and L2 are lattice digraphs (IsLatticeDigraph (6.4.3) returns true, then LatticeDigraphEmbedding returns a single injective DigraphHomomorphism (7.3.2) between L1 and L2, with the property that it is a lattice homomorphism. If no such homomorphism exists, fail is returned.

A lattice homomorphism is a digraph homomorphism which respects meets and joins of every pair of vertices. Note that every injective lattice homomorphism map is an embedding, in the sense that the inverse of map is a lattice homomorphism also.

Example

gap> D := DigraphReflexiveTransitiveClosure(ChainDigraph(5));
<immutable preorder digraph with 5 vertices, 15 edges>
gap> L1 := DigraphReflexiveTransitiveClosure(ChainDigraph(5));
<immutable preorder digraph with 5 vertices, 15 edges>
gap> L2 := DigraphReflexiveTransitiveClosure(ChainDigraph(6));
<immutable preorder digraph with 6 vertices, 21 edges>
gap> LatticeDigraphEmbedding(L1, L2);
IdentityTransformation
gap> LatticeDigraphEmbedding(L2, L1);
fail

7.3.20 IsLatticeHomomorphism (for digraphs and a permutation or transformation)

▷ IsLatticeHomomorphism(L1, L2, map) (operation)
▷ IsLatticeEpimorphism(L1, L2, map) (operation)
▷ IsLatticeEmbedding(L1, L2, map) (operation)
▷ IsLatticeMonomorphism(L1, L2, map) (operation)
▷ IsLatticeEndomorphism(L, map) (operation)

Returns: true or false.

Each of the function described in this section (except IsLatticeEndomorphism) takes a pair of digraphs L1 and L2, and a transformation map, returning true if map is a lattice homomorphism from L1 to L2, and false otherwise. If L1 or L2 is not a lattice, then false is returned.

A transformation or permutation map is a lattice homomorphism if map respects meets and joins of every pair of vertices, and map fixes every i which is not a vertex of L1.

IsLatticeHomomorphism returns true if the permutation or transformation map is a lattice homomorphism from the lattice digraph L1 to the lattice digraph L2.

IsLatticeEpimorphism returns true if the permutation or transformation map is a surjective lattice homomorphism from the lattice digraph L1 to the lattice digraph L2.
IsLatticeEmbedding returns true if the permutation or transformation map is an injective lattice homomorphism from the lattice digraph \( L_1 \) to the lattice digraph \( L_2 \). The function IsLatticeMonomorphism is a synonym of IsLatticeEmbedding.

IsLatticeEndomorphism returns true if the permutation or transformation map is an lattice endomorphism of the lattice digraph \( L \).

```
Example

gap> G := Digraph([[2, 4], [3, 7], [6], [5, 7], [6], [], [6]]);
<immutable digraph with 7 vertices, 9 edges>
gap> D := DigraphRemoveVertex(G, 7);
<immutable digraph with 6 vertices, 6 edges>
gap> G := DigraphReflexiveTransitiveClosure(G);
<immutable preorder digraph with 7 vertices, 22 edges>
gap> D := DigraphReflexiveTransitiveClosure(D);
<immutable preorder digraph with 6 vertices, 17 edges>
gap> IsDigraphEmbedding(D, G, IdentityTransformation);  # true

gap> D := Digraph([[2, 3], [4], [4], []]);
<immutable digraph with 4 vertices, 4 edges>
gap> G := Digraph([[2, 3], [4], [4], [5], []]);
<immutable digraph with 5 vertices, 5 edges>
gap> D := DigraphReflexiveTransitiveClosure(D);
<immutable preorder digraph with 4 vertices, 9 edges>
gap> G := DigraphReflexiveTransitiveClosure(G);
<immutable preorder digraph with 5 vertices, 14 edges>
gap> IsLatticeEmbedding(D, G, IdentityTransformation);  # true

gap> IsLatticeMonomorphism(D, G, IdentityTransformation);
true

gap> f := Transformation([[1, 2, 3, 4, 4]]);
Transformation( [ 1, 2, 3, 4, 4 ] )
gap> IsLatticeEpimorphism(G, D, f);  # true

gap> IsLatticeEndomorphism(D, (2, 3));  # true
```
Chapter 8

Finding cliques and independent sets

In Digraphs, a *clique* of a digraph is a set of mutually adjacent vertices of the digraph, and an *independent set* is a set of mutually non-adjacent vertices of the digraph. A *maximal clique* is a clique which is not properly contained in another clique, and a *maximal independent set* is an independent set which is not properly contained in another independent set. Using this definition in Digraphs, cliques and independent sets are both permitted, but not required, to contain vertices at which there is a loop. Another name for a clique is a *complete subgraph*.

Digraphs provides extensive functionality for computing cliques and independent sets of a digraph, whether maximal or not. The fundamental algorithm used in most of the methods in Digraphs to calculate cliques and independent sets is a version of the Bron-Kerbosch algorithm. Calculating the cliques and independent sets of a digraph is a well-known and hard problem, and searching for cliques or independent sets in a digraph can be very lengthy, even for a digraph with a small number of vertices. Digraphs uses several strategies to increase the performance of these calculations.

From the definition of cliques and independent sets, it follows that the presence of loops and multiple edges in a digraph is irrelevant to the existence of cliques and independent sets in the digraph. See DigraphHasLoops (6.2.1) and IsMultiDigraph (6.2.11) for more information about these properties. Therefore given a digraph $digraph$, the cliques and independent sets of $digraph$ are equal to the cliques and independent sets of the digraph:

- $DigraphRemoveLoops(DigraphRemoveAllMultipleEdges(digraph))$.

See DigraphRemoveLoops (3.3.25) and DigraphRemoveAllMultipleEdges (3.3.26) for more information about these attributes. Furthermore, the cliques of this digraph are equal to the cliques of the digraph formed by removing any edge $[u,v]$ for which the corresponding reverse edge $[v,u]$ is not present. Therefore, overall, the cliques of $digraph$ are equal to the cliques of the symmetric digraph:

- $MaximalSymmetricSubdigraphWithoutLoops(digraph)$.

See MaximalSymmetricSubdigraphWithoutLoops (3.3.5) for more information about this. The AutomorphismGroup (7.2.2) of this symmetric digraph contains the automorphism group of $digraph$ as a subgroup. By performing the search for maximal cliques with the help of this larger automorphism group to reduce the search space, the computation time may be reduced. The functions and attributes which return representatives of cliques of $digraph$ will return orbit representatives of cliques under the action of the automorphism group of the *maximal symmetric subdigraph without loops* on sets of vertices.

189
The independent sets of a digraph are equal to the independent sets of the DigraphSymmetricClosure (3.3.12). Therefore, overall, the independent sets of digraph are equal to the independent sets of the symmetric digraph:

- DigraphSymmetricClosure(DigraphRemoveLoops(DigraphRemoveAllMultipleEdges(digraph))).

Again, the automorphism group of this symmetric digraph contains the automorphism group of digraph. Since a search for independent sets is equal to a search for cliques in the DigraphDual (3.3.11), the methods used in Digraphs usually transform a search for independent sets into a search for cliques, as described above. The functions and attributes which return representatives of independent sets of digraph will return orbit representatives of independent sets under the action of the automorphism group of the symmetric closure of the digraph formed by removing loops and multiple edges.

Please note that in Digraphs, cliques and independent sets are not required to be maximal. Some authors use the word clique to mean maximal clique, and some authors use the phrase independent set to mean maximal independent set, but please note that Digraphs does not use this definition.

### 8.1 Finding cliques

#### 8.1.1 IsClique

▷ IsClique(digraph, l) (operation)
▷ IsMaximalClique(digraph, l) (operation)

**Returns:** true or false.

If digraph is a digraph and l is a duplicate-free list of vertices of digraph, then IsClique(digraph, l) returns true if l is a clique of digraph and false if it is not. Similarly, IsMaximalClique(digraph, l) returns true if l is a maximal clique of digraph and false if it is not.

A clique of a digraph is a set of mutually adjacent vertices of the digraph. A maximal clique is a clique that is not properly contained in another clique. A clique is permitted, but not required, to contain vertices at which there is a loop.

```
Example

gap> D := CompleteDigraph(4);
<immutable multidigraph with 4 vertices, 6 edges>

gap> IsClique(D, [1, 3, 2]);
true

gap> IsMaximalClique(D, [1, 3, 2]);
true

gap> IsMaximalClique(D, DigraphVertices(D));
true

gap> D := Digraph([[1, 2, 4, 4], [1, 3, 4], [2, 1], [1, 2]]);
<immutable multidigraph with 4 vertices, 11 edges>

gap> IsClique(D, [2, 3, 4]);
false

gap> IsMaximalClique(D, [1, 2, 4]);
true

gap> D := CompleteDigraph(IsMutableDigraph, 4);
<mutable multidigraph with 4 vertices, 6 edges>

gap> IsClique(D, [1, 3, 2]);
true
```

```
8.1.2 CliquesFinder

\[\text{CliquesFinder}(\text{digraph}, \text{hook}, \text{user\_param}, \text{limit}, \text{include}, \text{exclude}, \text{max}, \text{size}, \text{reps})\]

\[\text{(function)}\]

**Returns:** The argument \text{user\_param}.

This function finds cliques of the digraph \text{digraph} subject to the conditions imposed by the other arguments as described below. Note that a clique is represented by the immutable list of the vertices that it contains.

Let \(G\) denote the automorphism group of the maximal symmetric subdigraph of \text{digraph} without loops (see AutomorphismGroup (7.2.2) and MaximalSymmetricSubdigraphWithoutLoops (3.3.5)).

\text{hook}

This argument should be a function or \text{fail}.

If \text{hook} is a function, then it should have two arguments \text{user\_param} (see below) and a clique \(c\). The function \text{hook}(\text{user\_param}, c) is called every time a new clique \(c\) is found by CliquesFinder.

If \text{hook} is \text{fail}, then a default function is used that simply adds every new clique found by CliquesFinder to \text{user\_param}, which must be a list in this case.

\text{user\_param}

If \text{hook} is a function, then \text{user\_param} can be any \text{GAP} object. The object \text{user\_param} is used as the first argument for the function \text{hook}. For example, \text{user\_param} might be a list, and \text{hook}(\text{user\_param}, c) might add the size of the clique \(c\) to the list \text{user\_param}.

If the value of \text{hook} is \text{fail}, then the value of \text{user\_param} must be a list.

\text{limit}

This argument should be a positive integer or \text{infinity}. CliquesFinder will return after it has found \text{limit} cliques or the search is complete.

\text{include} and \text{exclude}

These arguments should each be a (possibly empty) duplicate-free list of vertices of \text{digraph} (i.e. positive integers less than the number of vertices of \text{digraph}).

CliquesFinder will only look for cliques containing all of the vertices in \text{include} and containing none of the vertices in \text{exclude}.

Note that the search may be much more efficient if each of these lists is invariant under the action of \(G\) on sets of vertices.

\text{max}

This argument should be \text{true} or \text{false}. If \text{max} is \text{true} then CliquesFinder will only search for \text{maximal} cliques. If \text{max} is \text{false} then non-maximal cliques may be found.

\text{size}

This argument should be \text{fail} or a positive integer. If \text{size} is a positive integer then CliquesFinder will only search for cliques that contain precisely \text{size} vertices. If \text{size} is \text{fail} then cliques of any size may be found.

\text{reps}

This argument should be \text{true} or \text{false}.
If `reps` is `true` then the arguments `include` and `exclude` are each required to be invariant under the action of G on sets of vertices. In this case, `CliquesFinder` will find representatives of the orbits of the desired cliques under the action of G, although representatives may be returned that are in the same orbit. If `reps` is `false` then `CliquesFinder` will not take this into consideration.

For a digraph such that G is non-trivial, the search for clique representatives can be much more efficient than the search for all cliques.

This function uses a version of the Bron-Kerbosch algorithm.

Example

```plaintext
gap> D := CompleteDigraph(5);  # Immutable complete digraph with 5 vertices
<immutable complete digraph with 5 vertices>

gap> user_param := [];;

gap> f := function(a, b)  # Calculate size of clique
> AddSet(user_param, Size(b));
> end;;

gap> CliquesFinder(D, f, user_param, infinity, [], [], false, fail, true);
[ 1, 2, 3, 4, 5 ]

gap> CliquesFinder(D, fail, [], 5, [2, 4], [3], false, fail, false);
[ [ 2, 4 ], [ 1, 2, 4 ], [ 2, 4, 5 ], [ 1, 2, 4, 5 ] ]

gap> CliquesFinder(D, fail, [], 2, [2, 4], [3], false, fail, false);
[ [ 2, 4 ], [ 1, 2, 4 ] ]

gap> CliquesFinder(D, fail, [], infinity, [], [], true, 5, false);
[ 1, 2, 3, 4, 5 ]

gap> CliquesFinder(D, fail, [], infinity, [1, 3], [], false, 3, false);
[ [ 1, 2, 3 ], [ 1, 3, 4 ], [ 1, 3, 5 ] ]

gap> CliquesFinder(D, fail, [], infinity, [1, 3], [], true, 3, false);
[ ]

gap> D := CompleteDigraph(IsMutableDigraph, 5);  # Mutable digraph with 5 vertices, 20 edges
<mutable digraph with 5 vertices, 20 edges>

gap> user_param := [];;

gap> f := function(a, b)  # Calculate size of clique
> AddSet(user_param, Size(b));
> end;;

gap> CliquesFinder(D, f, user_param, infinity, [], [], false, fail, true);
[ 1, 2, 3, 4, 5 ]

gap> CliquesFinder(D, fail, [], 5, [2, 4], [3], false, fail, false);
[ [ 2, 4 ], [ 1, 2, 4 ], [ 2, 4, 5 ], [ 1, 2, 4, 5 ] ]

gap> CliquesFinder(D, fail, [], 2, [2, 4], [3], false, fail, false);
[ [ 2, 4 ], [ 1, 2, 4 ] ]

gap> CliquesFinder(D, fail, [], infinity, [], [], true, 5, false);
[ 1, 2, 3, 4, 5 ]

gap> CliquesFinder(D, fail, [], infinity, [1, 3], [], false, 3, false);
[ [ 1, 2, 3 ], [ 1, 3, 4 ], [ 1, 3, 5 ] ]

gap> CliquesFinder(D, fail, [], infinity, [1, 3], [], true, 3, false);
[ ]
```
8.1.3 DigraphClique

\[\text{DigraphClique}(\text{digraph}[, \text{include}[\text{, exclude}[\text{, size}]]])\]  (function)
\[\text{DigraphMaximalClique}(\text{digraph}[, \text{include}[\text{, exclude}[\text{, size}]]])\]  (function)

Returns: An immutable list of positive integers, or fail.

If \text{digraph} is a digraph, then these functions returns a clique of \text{digraph} if one exists that satisfies the arguments, else it returns fail. A clique is defined by the set of vertices that it contains; see \text{IsClique} (8.1.1) and \text{IsMaximalClique} (8.1.1).

The optional arguments \text{include} and \text{exclude} must each be a (possibly empty) duplicate-free list of vertices of \text{digraph}, and the optional argument \text{size} must be a positive integer. By default, \text{include} and \text{exclude} are empty. These functions will search for a clique of \text{digraph} that includes the vertices of \text{include} but does not include any vertices in \text{exclude}; if the argument \text{size} is supplied, then additionally the clique will be required to contain precisely \text{size} vertices.

If \text{include} is not a clique, then these functions return fail. Otherwise, the functions behave in the following way, depending on the number of arguments:

One or two arguments

If one or two arguments are supplied, then \text{DigraphClique} and \text{DigraphMaximalClique} greedily enlarge the clique \text{include} until it cannot continue. The result is guaranteed to be a maximal clique. This may or may not return an answer more quickly than using \text{DigraphMaximalCliques} (8.1.4) with a limit of 1.

Three arguments

If three arguments are supplied, then \text{DigraphClique} greedily enlarges the clique \text{include} until it cannot continue, although this clique may not be maximal.

Given three arguments, \text{DigraphMaximalClique} returns the maximal clique returned by \text{DigraphMaximalCliques}(\text{digraph}, \text{include}, \text{exclude}, 1) if one exists, else fail.

Four arguments

If four arguments are supplied, then \text{DigraphClique} returns the clique returned by \text{DigraphCliques}(\text{digraph}, \text{include}, \text{exclude}, 1, \text{size}) if one exists, else fail. This clique may not be maximal.

Given four arguments, \text{DigraphMaximalClique} returns the maximal clique returned by \text{DigraphMaximalCliques}(\text{digraph}, \text{include}, \text{exclude}, 1, \text{size}) if one exists, else fail.

Example

\begin{verbatim}
gap> D := Digraph(([2, 3, 4], [1, 3], [1, 2], [1, 5], []));
<immutable digraph with 5 vertices, 9 edges>
gap> IsSymmetricDigraph(D);
false
gap> DigraphClique(D); [ 5 ]
gap> DigraphMaximalClique(D); [ 5 ]
gap> DigraphClique(D, [1, 2]); [ 1, 2, 3 ]
gap> DigraphMaximalClique(D, [1, 3]); [ 1, 3, 2 ]
gap> DigraphClique(D, [1], [2]);
\end{verbatim}
8.1.4 DigraphMaximalCliques

▷ DigraphMaximalCliques(digraph[, include[, exclude[, limit[, size]]]])
▷ DigraphMaximalCliquesReps(digraph[, include[, exclude[, limit[, size]]]])
▷ DigraphCliques(digraph[, include[, exclude[, limit[, size]]]])
▷ DigraphCliquesReps(digraph[, include[, exclude[, limit[, size]]]])
▷ DigraphMaximalCliquesAttr(digraph)
▷ DigraphMaximalCliquesRepsAttr(digraph)

Returns: An immutable list of lists of positive integers.

If digraph is digraph, then these functions and attributes use CliquesFinder (8.1.2) to return cliques of digraph. A clique is defined by the set of vertices that it contains; see IsClique (8.1.1) and IsMaximalClique (8.1.1).

The optional arguments include and exclude must each be a (possibly empty) list of vertices of digraph, the optional argument limit must be either a positive integer or infinity, and the optional argument size must be a positive integer. If not specified, then include and exclude are chosen to be empty lists, and limit is set to infinity.

The functions will return as many suitable cliques as possible, up to the number limit. These functions will find cliques that contain all of the vertices of include but do not contain any of the vertices of exclude. The argument size restricts the search to those cliques that contain precisely size vertices. If the function or attribute has Maximal in its name, then only maximal cliques will be returned; otherwise non-maximal cliques may be returned.

Let G denote the automorphism group of maximal symmetric subdigraph of digraph without loops (see AutomorphismGroup (7.2.2) and MaximalSymmetricSubdigraphWithoutLoops (3.3.5)).

Distinct cliques

DigraphMaximalCliques and DigraphCliques each return a duplicate-free list of at most limit cliques of digraph that satisfy the arguments.

The computation may be significantly faster if include and exclude are invariant under the action of G on sets of vertices.

Orbit representatives of cliques

To use DigraphMaximalCliquesReps or DigraphCliquesReps, the arguments include and
exclude must each be invariant under the action of $G$ on sets of vertices.

If this is the case, then DigraphMaximalCliquesReps and DigraphCliquesReps each return a duplicate-free list of at most limit orbits representatives (under the action of $G$ on sets vertices) of cliques of digraph that satisfy the arguments.

The representatives are not guaranteed to be in distinct orbits. However, if fewer than $\lim$ results are returned, then there will be at least one representative from each orbit of maximal cliques.

Example

```
gap> D := Digraph([ 
> [2, 3], [1, 3], [1, 2, 4], [3, 5, 6], [4, 6], [4, 5]]); 
<immutable digraph with 6 vertices, 14 edges>
gap> IsSymmetricDigraph(D); 
true 
gap> G := AutomorphismGroup(D); 
Group([ (5,6), (1,2), (1,5)(2,6)(3,4) ]) 
gap> DigraphMaximalCliques(D); 
[ [ 1, 2, 3 ], [ 4, 5, 6 ], [ 3, 4 ] ] 
gap> DigraphMaximalCliquesReps(D); 
[ [ 1, 2, 3 ], [ 3, 4 ] ] 
gap> Orbit(G, [1, 2, 3], OnSets); 
[ [ 1, 2, 3 ], [ 4, 5, 6 ] ] 
gap> Orbit(G, [3, 4], OnSets); 
[ [ 3, 4 ] ] 
gap> DigraphMaximalCliquesReps(D, [3, 4], [], 1); 
[ [ 3, 4 ] ] 
gap> DigraphMaximalCliquesReps(D, [3, 4], [1], 1); 
[ [ 3, 4 ] ] 
gap> DigraphMaximalCliques(D, [1, 2], [5, 6], 1, 2); 
[ ] 
gap> DigraphCliques(D, [1], [5, 6], infinity, 2); 
[ [ 1, 2 ], [ 1, 3 ] ] 
gap> D := Digraph(IsMutableDigraph, [ 
> [2, 3], [1, 3], [1, 2, 4], [3, 5, 6], [4, 6], [4, 5]]); 
<mutable digraph with 6 vertices, 14 edges>
gap> IsSymmetricDigraph(D); 
true 
gap> G := AutomorphismGroup(D); 
Group([ (5,6), (1,2), (1,5)(2,6)(3,4) ]) 
gap> DigraphMaximalCliques(D); 
[ [ 1, 2, 3 ], [ 4, 5, 6 ], [ 3, 4 ] ]
```

8.1.5 CliqueNumber

▷ CliqueNumber(digraph) (attribute)

Returns: A non-negative integer.

If digraph is a digraph, then CliqueNumber(digraph) returns the largest integer $n$ such that digraph contains a clique with $n$ vertices as an induced subdigraph.

A clique of a digraph is a set of mutually adjacent vertices of the digraph. Loops and multiple edges are ignored for the purpose of determining the clique number of a digraph.

Example

```
gap> D := CompleteDigraph(4); 
gap> CliqueNumber(D); 
```
Digraphs

4
gap> D := Digraph([[1, 2, 4, 4], [1, 3, 4], [2, 1], [1, 2]]);
<immutable multidigraph with 4 vertices, 11 edges>
gap> CliqueNumber(D);
3
gap> D := CompleteDigraph(IsMutableDigraph, 4);

8.2 Finding independent sets

8.2.1 IsIndependentSet

▷ IsIndependentSet(digraph, l) (operation)
▷ IsMaximalIndependentSet(digraph, l) (operation)

Returns: true or false.

If digraph is a digraph and l is a duplicate-free list of vertices of digraph, then IsIndependentSet(digraph, l) returns true if l is an independent set of digraph and false if it is not. Similarly, IsMaximalIndependentSet(digraph, l) returns true if l is a maximal independent set of digraph and false if it is not.

An independent set of a digraph is a set of mutually non-adjacent vertices of the digraph. A maximal independent set is an independent set that is not properly contained in another independent set. An independent set is permitted, but not required, to contain vertices at which there is a loop.

Example

8.2.2 DigraphIndependentSet

▷ DigraphIndependentSet(digraph[, include[, exclude[, size]]]) (function)
▷ DigraphMaximalIndependentSet(digraph[, include[, exclude[, size]]]) (function)

Returns: An immutable list of positive integers, or fail.

If digraph is a digraph, then these functions returns an independent set of digraph if one exists that satisfies the arguments, else it returns fail. An independent set is defined by the set of vertices that it contains; see IsIndependentSet (8.2.1) and IsMaximalIndependentSet (8.2.1).

The optional arguments include and exclude must each be a (possibly empty) duplicate-free list of vertices of digraph, and the optional argument size must be a positive integer. By default,
include and exclude are empty. These functions will search for an independent set of digraph that includes the vertices of include but does not include any vertices in exclude; if the argument size is supplied, then additionally the independent set will be required to contain precisely size vertices.

If include is not an independent set, then these functions return fail. Otherwise, the functions behave in the following way, depending on the number of arguments:

**One or two arguments**

If one or two arguments are supplied, then DigraphIndependentSet and DigraphMaximalIndependentSet greedily enlarge the independent set include until it cannot continue. The result is guaranteed to be a maximal independent set. This may or may not return an answer more quickly than using DigraphMaximalIndependentSets (8.2.3) with a limit of 1.

**Three arguments**

If three arguments are supplied, then DigraphIndependentSet greedily enlarges the independent set include until it cannot continue, although this independent set may not be maximal.

Given three arguments, DigraphMaximalIndependentSet returns the maximal independent set returned by DigraphMaximalIndependentSets(digraph, include, exclude, 1) if one exists, else fail.

**Four arguments**

If four arguments are supplied, then DigraphIndependentSet returns the independent set returned by DigraphIndependentSets(digraph, include, exclude, 1, size) if one exists, else fail. This independent set may not be maximal.

Given four arguments, DigraphMaximalIndependentSet returns the maximal independent set returned by DigraphMaximalIndependentSets(digraph, include, exclude, 1, size) if one exists, else fail.

---

**Example**

```gap
gap> D := ChainDigraph(6);
<immutable chain digraph with 6 vertices>
gap> DigraphIndependentSet(D);
[ 6, 4, 2 ]
gap> DigraphMaximalIndependentSet(D);
[ 6, 4, 2 ]
gap> DigraphIndependentSet(D, [2, 4]);
[ 2, 4, 6 ]
gap> DigraphMaximalIndependentSet(D, [1, 3]);
[ 1, 3, 6 ]
gap> DigraphIndependentSet(D, [2, 4], [6]);
[ 2, 4 ]
gap> DigraphMaximalIndependentSet(D, [2, 4], [6]);
fail
gap> DigraphIndependentSet(D, [1], [], 2);
[ 1, 3 ]
gap> DigraphMaximalIndependentSet(D, [1], [], 2);
fail
gap> DigraphMaximalIndependentSet(D, [1], [], 3);
[ 1, 3, 5 ]
gap> D := ChainDigraph(IsMutableDigraph, 6);
<mutable digraph with 6 vertices, 5 edges>
```
8.2.3 DigraphMaximalIndependentSets

- DigraphMaximalIndependentSets(digraph[, include[, exclude[, limit[, size]]]]) (function)
- DigraphMaximalIndependentSetsReps(digraph[, include[, exclude[, limit[, size]]]]) (function)
- DigraphIndependentSets(digraph[, include[, exclude[, limit[, size]]]]) (function)
- DigraphIndependentSetsReps(digraph[, include[, exclude[, limit[, size]]]]) (function)
- DigraphMaximalIndependentSetsAttr(digraph) (attribute)
- DigraphMaximalIndependentSetsRepsAttr(digraph) (attribute)

**Returns:** An immutable list of lists of positive integers.

If `digraph` is digraph, then these functions and attributes use CliquesFinder (8.1.2) to return independent sets of `digraph`. An independent set is defined by the set of vertices that it contains; see IsMaximalIndependentSet (8.2.1) and IsIndependentSet (8.2.1).

The optional arguments `include` and `exclude` must each be a (possibly empty) list of vertices of `digraph`, the optional argument `limit` must be either a positive integer or infinity, and the optional argument `size` must be a positive integer. If not specified, then `include` and `exclude` are chosen to be empty lists, and `limit` is set to infinity.

The functions will return as many suitable independent sets as possible, up to the number `limit`. These functions will find independent sets that contain all of the vertices of `include` but do not contain any of the vertices of `exclude`. The argument `size` restricts the search to those cliques that contain precisely `size` vertices. If the function or attribute has `Maximal` in its name, then only maximal independent sets will be returned; otherwise non-maximal independent sets may be returned.

Let G denote the AutomorphismGroup (7.2.2) of the DigraphSymmetricClosure (3.3.12) of the digraph formed from `digraph` by removing loops and ignoring the multiplicity of edges.

**Distinct independent sets**

DigraphMaximalIndependentSets and DigraphIndependentSets each return a duplicate-
free list of at most \texttt{limit} independent sets of \texttt{digraph} that satisfy the arguments.

The computation may be significantly faster if \texttt{include} and \texttt{exclude} are invariant under the action of \texttt{G} on sets of vertices.

**Representatives of distinct orbits of independent sets**

To use \texttt{DigraphMaximalIndependentSetsReps} or \texttt{DigraphIndependentSetsReps}, the arguments \texttt{include} and \texttt{exclude} must each be invariant under the action of \texttt{G} on sets of vertices.

If this is the case, then \texttt{DigraphMaximalIndependentSetsReps} and \texttt{DigraphIndependentSetsReps} each return a list of at most \texttt{limit} orbits representatives (under the action of \texttt{G} on sets of vertices) of independent sets of \texttt{digraph} that satisfy the arguments.

The representatives are not guaranteed to be in distinct orbits. However, if \texttt{lim} is not specified, or fewer than \texttt{lim} results are returned, then there will be at least one representative from each orbit of maximal independent sets.

---

**Example**

```gap
D := CycleDigraph(5);
<immutable cycle digraph with 5 vertices>
DigraphMaximalIndependentSetsReps(D);
[ [ 1, 3 ] ]
DigraphIndependentSetsReps(D);
[ [ 1 ], [ 1, 3 ] ]
Set(DigraphMaximalIndependentSets(D));
[ [ 1, 3 ], [ 1, 4 ], [ 2, 4 ], [ 2, 5 ], [ 3, 5 ] ]
DigraphMaximalIndependentSets(D, [1]);
[ [ 1, 3 ], [ 1, 4 ] ]
DigraphIndependentSets(D, [], [4, 5]);
[ [ 1 ], [ 2 ], [ 3 ], [ 1, 3 ] ]
DigraphIndependentSets(D, [], [4, 5], 1, 2);
[ [ 1, 3 ] ]
D := CycleDigraph(IsMutableDigraph, 5);
<mutable digraph with 5 vertices, 5 edges>
DigraphMaximalIndependentSetsReps(D);
[ [ 1, 3 ] ]
DigraphIndependentSetsReps(D);
[ [ 1 ], [ 1, 3 ] ]
Set(DigraphMaximalIndependentSets(D));
[ [ 1, 3 ], [ 1, 4 ], [ 2, 4 ], [ 2, 5 ], [ 3, 5 ] ]
DigraphMaximalIndependentSets(D, [1]);
[ [ 1, 3 ], [ 1, 4 ] ]
DigraphIndependentSets(D, [], [4, 5]);
[ [ 1 ], [ 2 ], [ 3 ], [ 1, 3 ] ]
DigraphIndependentSets(D, [], [4, 5], 1, 2);
[ [ 1, 3 ] ]
```
Chapter 9

Visualising and IO

9.1 Visualising a digraph

9.1.1 Splash

\texttt{\textbackslash{}Splash\texttt{(str[, opts])}}

\textbf{Returns:} Nothing.

This function attempts to convert the string \texttt{str} into a pdf document and open this document, i.e. to splash it all over your monitor.

The string \texttt{str} must correspond to a valid \texttt{dot} or \LaTeX{} text file and you must have have GraphViz and \texttt{pdflatex} installed on your computer. For details about these file formats, see \url{https://www.latex-project.org} and \url{https://www.graphviz.org}.

This function is provided to allow convenient, immediate viewing of the pictures produced by the function \texttt{DotDigraph (9.1.2)}.

The optional second argument \texttt{opts} should be a record with components corresponding to various options, given below.

\textbf{path} this should be a string representing the path to the directory where you want \texttt{Splash} to do its work. The default value of this option is "~/".

\textbf{directory} this should be a string representing the name of the directory in \texttt{path} where you want \texttt{Splash} to do its work. This function will create this directory if does not already exist.

The default value of this option is "tmp.viz" if the option \texttt{path} is present, and the result of \texttt{DirectoryTemporary (Reference: DirectoryTemporary)} is used otherwise.

\textbf{filename} this should be a string representing the name of the file where \texttt{str} will be written. The default value of this option is "vizpicture".

\textbf{viewer} this should be a string representing the name of the program which should open the files produced by GraphViz or \texttt{pdflatex}.

\textbf{type} this option can be used to specify that the string \texttt{str} contains a \texttt{\LaTeX} or \texttt{dot} document. You can specify this option in \texttt{str} directly by making the first line "\%latex" or "/\dot". There is no default value for this option, this option must be specified in \texttt{str} or in \texttt{opt.type}. 

200
engine

this option can be used to specify the GraphViz engine to use to render a dot document. The valid choices are "dot", "neato", "circo", "twopi", "fdp", "sfdp", and "patchwork". Please refer to the GraphViz documentation for details on these engines. The default value for this option is "dot", and it must be specified in opt.engine.

filetype

this should be a string representing the type of file which Splash should produce. For \LaTeX files, this option is ignored and the default value "pdf" is used.

This function was written by Attila Egri-Nagy and Manuel Delgado with some minor changes by J. D. Mitchell.

Example

\begin{verbatim}
gap> Splash(DotDigraph(RandomDigraph(4))); \end{verbatim}

9.1.2 DotDigraph

\begin{verbatim}
DotDigraph(digraph) \hspace{1cm} (attribute)
DotColoredDigraph(digraph, vert, edge) \hspace{1cm} (operation)
DotVertexLabelledDigraph(digraph) \hspace{1cm} (operation)
DotVertexColoredDigraph(digraph, vert) \hspace{1cm} (operation)
DotEdgeColoredDigraph(digraph, edge) \hspace{1cm} (operation)
\end{verbatim}

Returns: A string.

DotDigraph produces a graphical representation of the digraph \texttt{digraph}. Vertices are displayed as circles, numbered consistently with \texttt{digraph}. Edges are displayed as arrowed lines between vertices, with the arrowhead of each line pointing towards the range of the edge.

DotColoredDigraph differs from DotDigraph only in that the values in given in the two lists are used to color the vertices and edges of the graph when displayed. The list for vertex colours should be a list of length equal to the number of vertices, containing strings that are accepted by the graphviz software, which is the one used for graph representation. The list for edge colours should be a list of lists with the same shape of the outneighbours of the digraph that contains strings that correspond to colours accepted by the graphviz software. If the lists are not the appropriate size, or have holes then the function will return an error.

DotVertexColoredDigraph differs from DotDigraph only in that the values in given in the list are used to color the vertices of the graph when displayed. The list for vertex colours should be a list of length equal to the number of vertices, containing strings that are accepted by the graphviz software, which is the one used for graph representation. If the list is not the appropriate size, or has holes then the function will return an error.

DotEdgeColoredDigraph differs from DotDigraph only in that the values in given in the list are used to color the vertices and edges of the graph when displayed. The list for edge colours should be a list of lists with the same shape of the outneighbours of the digraph that contains strings that correspond to colours accepted by the graphviz software. If the list is not the appropriate size, or has holes then the function will return an error.

DotVertexLabelledDigraph differs from DotDigraph only in that the values in given in the list are used to label the vertices in the produced picture rather than the numbers 1 to the number of vertices of the digraph.

The output is in dot format (also known as GraphViz) format. For details about this file format, and information about how to display or edit this format see https://www.graphviz.org.
The string returned by DotDigraph or DotVertexLabelledDigraph can be written to a file using the command FileString (GAPDoc: FileString).

Example

```gap
gap> adj := List([1 .. 4], x -> [1, 1, 1, 1]);
[ [ 1, 1, 1, 1 ], [ 1, 1, 1, 1 ], [ 1, 1, 1, 1 ], [ 1, 1, 1, 1 ] ]
gap> adj[1][3] := 0;
0
gap> gr := DigraphByAdjacencyMatrix(adj);
<immutable digraph with 4 vertices, 15 edges>
gap> D := CompleteDigraph(4);
<immutable complete digraph with 4 vertices>
gap> vertcolors := [];;
gap> edgecolors := [];;
gap> edgecolors[1][2] := "lightblue";;
gap> edgecolors[1][3] := "pink";;
gap> edgecolors[1][4] := "purple";;
gap> edgecolors[2][1] := "lightblue";;
gap> edgecolors[2][3] := "pink";;
gap> edgecolors[2][4] := "purple";;
gap> edgecolors[3][1] := "lightblue";;
gap> edgecolors[3][2] := "pink";;
gap> edgecolors[3][4] := "purple";;
gap> edgecolors[4][1] := "lightblue";;
gap> edgecolors[4][2] := "pink";;
gap> edgecolors[4][3] := "purple";;
gap> Print(DotColoredDigraph(D, vertcolors, edgecolors));
//dot
digraph hgn{
  node [shape=circle]
  1[color=blue, style=filled]
  2[color=red, style=filled]
  3[color=green, style=filled]
  4[color=yellow, style=filled]
  1 -> 2[color=lightblue]
  1 -> 3[color=pink]
  1 -> 4[color=yellow]
  2 -> 1[color=lightblue]
  2 -> 3[color=pink]
  2 -> 4[color=yellow]
  3 -> 1[color=lightblue]
  3 -> 2[color=pink]
  3 -> 4[color=yellow]
  4 -> 1[color=lightblue]
  4 -> 2[color=pink]
  4 -> 3[color=yellow]
}
gap> D := EmptyDigraph(3);
<immutable empty digraph with 3 vertices>
gap> vertcolors := [];;
```
Digraphs

```gap
gap> vertcolors[3] := "green";;
gap> edgecolors := [];;
gap> Print(DotColoredDigraph(D, vertcolors, edgecolors));
//dot
digraph hgn{
    node [shape=circle]
    1[color=blue, style=filled]
    2[color=red, style=filled]
    3[color=green, style=filled]
}
gap> D := Digraph([[2], [1, 3], [2]]);
<immutable digraph with 3 vertices, 4 edges>
gap> vertcolors := [];;
gap> vertcolors[1] := "blue";;
gap> vertcolors[2] := "pink";;
gap> vertcolors[3] := "purple";;
gap> edgecolors := [];;
gap> Print(DotSymmetricColoredDigraph(D, vertcolors, edgecolors));
//dot
digraph hgn{
    node [shape=circle]
    1[color=blue, style=filled]
    2[color=pink, style=filled]
    3[color=purple, style=filled]
    1 -- 2[color=green]
    2 -- 3[color=red]
}
gap> D := Digraph([[2, 3], [1, 3], [1]]);
<immutable digraph with 3 vertices, 5 edges>
gap> vertcolors := [];;
gap> vertcolors[3] := "green";;
gap> edgecolors := [];;
gap> edgecolors[1][2] := "orange";; edgecolors[1][3] := "yellow";;
gap> edgecolors[3][1] := "yellow";;
gap> Print(DotColoredDigraph(D, vertcolors, edgecolors));
//dot
digraph hgn{
    node [shape=circle]
    1[color=blue, style=filled]
    2[color=red, style=filled]
```
Digraphs

3[color=green, style=filled]
1 -> 2[color=orange]
1 -> 3[color=yellow]
2 -> 1[color=orange]
2 -> 3[color=pink]
3 -> 1[color=yellow]
}
gap> D := Digraph(IsMutableDigraph, [[2, 3], [1, 3], [1]]);
<mutable digraph with 3 vertices, 5 edges>
gap> vertcolors := [];;
gap> vertcolors[3] := "green";;
gap> edgecolors := [];;
gap> edgecolors[1][2] := "orange";; edgecolors[1][3] := "yellow";;
gap> edgecolors[3][1] := "yellow";;
gap> Print(DotColoredDigraph(D, vertcolors, edgecolors));
//dot
digraph hgn{
node [shape=circle]
1[color=blue, style=filled]
2[color=red, style=filled]
3[color=green, style=filled]
1 -> 2[color=orange]
1 -> 3[color=yellow]
2 -> 1[color=orange]
2 -> 3[color=pink]
3 -> 1[color=yellow]
}
gap> D;
<mutable digraph with 3 vertices, 5 edges>
gap> DotSymmetricDigraph(gr2){{[12 .. 70]};
" hgn{nnode [shape=circle]n1
n2
n3
n4
n1 -- 2
n2 -- 3
n3 -- 3
n3 -- "
gap> DotSymmetricDigraph(gr1);
Error, the argument <D> must be a symmetric digraph,
gap> D := CompleteDigraph(4);
<immutable complete digraph with 4 vertices>
gap> vertcolors := [];;
gap> Print(DotVertexColoredDigraph(D, vertcolors));
//dot
digraph hgn{
node [shape=circle]
1[color=blue, style=filled]
2[color=red, style=filled]
3[color=green, style=filled]
4[color=yellow, style=filled]
1 -> 2
1 -> 3
Digraphs

1 -> 4
2 -> 1
2 -> 3
2 -> 4
3 -> 1
3 -> 2
3 -> 4
4 -> 1
4 -> 2
4 -> 3

gap> D := CompleteDigraph(4);
<immutable complete digraph with 4 vertices>

gap> edgecolors := [];

gap> edgecolors[1] := [];


gap> edgecolors[1][2] := "lightblue";

gap> edgecolors[1][3] := "pink";

gap> edgecolors[1][4] := "purple";

gap> edgecolors[2][1] := "lightblue";

gap> edgecolors[2][3] := "pink";

gap> edgecolors[2][4] := "purple";

gap> edgecolors[3][1] := "lightblue";

gap> edgecolors[3][2] := "pink";

gap> edgecolors[3][4] := "purple";

gap> edgecolors[4][1] := "lightblue";

gap> edgecolors[4][2] := "pink";

gap> edgecolors[4][3] := "purple";

gap> Print(DotEdgeColoredDigraph(D, edgecolors));

//dot
digraph hgn{
node [shape=circle]
1
2
3
4
1 -> 2[color=lightblue]
1 -> 3[color=pink]
1 -> 4[color=white]
2 -> 1[color=lightblue]
2 -> 3[color=pink]
2 -> 4[color=white]
3 -> 1[color=lightblue]
3 -> 2[color=pink]
3 -> 4[color=white]
4 -> 1[color=lightblue]
4 -> 2[color=pink]
4 -> 3[color=white]
}
gap> FileString("dot/k4.dot", DotDigraph(gr));
154
9.1.3 DotSymmetricDigraph

▷ DotSymmetricDigraph(digraph)  
▷ DotSymmetricColoredDigraph(digraph, vert, edge)  
▷ DotSymmetricVertexColoredDigraph(digraph, vert)  
▷ DotSymmetricEdgeColoredDigraph(digraph, edge)

Returns: A string.

This function produces a graphical representation of the symmetric digraph digraph. DotSymmetricDigraph will return an error if digraph is not a symmetric digraph. See IsSymmetricDigraph (6.2.14).

The function DotSymmetricColoredDigraph differs from DotDigraph only in that the values given in the two lists are used to color the vertices and edges of the graph when displayed. The list for vertex colours should be a list of length equal to the number of vertices, containing strings that are accepted by the graphviz software, which is the one used for graph representation. The list for edge colours should be a list of lists with the same shape of the outneighbours of the digraph that contains strings that correspond to colours accepted by the graphviz software. If the list is not the appropriate size, or has holes then the function will return an error.

The function DotSymmetricVertexColoredDigraph differs from DotDigraph only in that the values in given in the list is used to color the vertices of the graph when displayed. The list for vertex colours should be a list of length equal to the number of vertices, containing strings that are accepted by the graphviz software, which is the one used for graph representation. If the list is not the appropriate size, or has holes then the function will return an error.

The function DotSymmetricEdgeColoredDigraph differs from DotDigraph only in that the values given in the list are used to color the edges of the graph when displayed. The list for edge colours should be a list of lists with the same shape of the outneighbours, containing strings that are accepted by the graphviz software, which is the one used for graph representation. If the list is not the appropriate size, or has holes then the function will return an error.

Vertices are displayed as circles, numbered consistently with digraph. Since digraph is symmetric, for every non-loop edge there is a complementary edge with opposite source and range. DotSymmetricDigraph displays each pair of complementary edges as a single line between the relevant vertices, with no arrowhead.

The output is in dot format (also known as GraphViz) format. For details about this file format, and information about how to display or edit this format see https://www.graphviz.org.

The string returned by DotSymmetricDigraph can be written to a file using the command FileString (GAPDoc: FileString).

Example

```
gap> star := Digraph([[2, 2, 3, 4], [1, 1], [1], [1, 4]]);
<immutable multidigraph with 4 vertices, 9 edges>
gap> IsSymmetricDigraph(star);
true
gap> FileString("dot/star.dot", DotSymmetricDigraph(star));
<immutable digraph with 3 vertices, 4 edges>
gap> vertcolors := [];
gap> vertcolors[1] := "blue";
gap> vertcolors[3] := "purple";
gap> edgecolors := [];
gap> edgecolors[1] := [];
```
Digraphs

```gap
gap> Print(DotSymmetricColoredDigraph(D, vertcolors, edgecolors));
//dot
graph hgn{
node [shape=circle]
1[color=blue, style=filled]
2[color=pink, style=filled]
3[color=purple, style=filled]
1 -- 2[color=green]
2 -- 3[color=red]
}
gap> D := Digraph([[2], [1, 3], [2]]);
<immutable digraph with 3 vertices, 4 edges>
gap> vertcolors := [];;
gap> vertcolors[1] := "blue";;
gap> vertcolors[2] := "pink";;
gap> vertcolors[3] := "purple";;
gap> Print(DotSymmetricVertexColoredDigraph(D, vertcolors));
//dot
graph hgn{
node [shape=circle]
1[color=blue, style=filled]
2[color=pink, style=filled]
3[color=purple, style=filled]
1 -- 2
2 -- 3
}
gap> D := Digraph([[2], [1, 3], [2]]);
<immutable digraph with 3 vertices, 4 edges>
gap> edgecolors := [];;
gap> Print(DotSymmetricEdgeColoredDigraph(D, edgecolors));
//dot
graph hgn{
node [shape=circle]
1
2
3
1 -- 2[color=green]
2 -- 3[color=red]
}
83
```
9.1.4 DotPartialOrderDigraph

▷ DotPartialOrderDigraph(digraph)

Returns: A string.

This function produces a graphical representation of a partial order digraph digraph. DotPartialOrderDigraph will return an error if digraph is not a partial order digraph. See IsPartialOrderDigraph (6.4.2).

Since digraph is a partial order, it is both reflexive and transitive. The output of DotPartialOrderDigraph is the DotDigraph (9.1.2) of the DigraphReflexiveTransitiveReduction (3.3.14) of digraph.

The output is in dot format (also known as GraphViz) format. For details about this file format, and information about how to display or edit this format see https://www.graphviz.org.

The string returned by DotPartialOrderDigraph can be written to a file using the command FileString (GAPDoc: FileString).

Example

```
gap> poset := Digraph([[1, 4], [2], [2, 3, 4], [4]]);
gap> IsPartialOrderDigraph(gr);
true
gap> FileString("dot/poset.dot", DotPartialOrderDigraph(gr));
83
```

9.1.5 DotPreorderDigraph

▷ DotPreorderDigraph(digraph)
▷ DotQuasiorderDigraph(digraph)

Returns: A string.

This function produces a graphical representation of a preorder digraph digraph. DotPreorderDigraph will return an error if digraph is not a preorder digraph. See IsPreorderDigraph (6.4.1).

A preorder digraph is reflexive and transitive but in general it is not anti-symmetric and may have strongly connected components containing more than one vertex. The QuotientDigraph (3.3.9) Q obtained by forming the quotient of digraph by the partition of its vertices into the strongly connected components satisfies IsPartialOrderDigraph (6.4.2). Thus every vertex of Q corresponds to a strongly connected component of digraph. The output of DotPreorderDigraph displays the DigraphReflexiveTransitiveReduction (3.3.14) of Q with vertices displayed as rounded rectangles labelled by all of the vertices of digraph in the corresponding strongly connected component.

The output is in dot format (also known as GraphViz) format. For details about this file format, and information about how to display or edit this format see https://www.graphviz.org.

The string returned by DotPreorderDigraph can be written to a file using the command FileString (GAPDoc: FileString).

Example

```
gap> preset := Digraph([[1, 2, 4, 5], [1, 2, 4, 5], [3, 4], [4], [1, 2, 4, 5]]);
gap> IsPreorderDigraph(gr);
true
gap> FileString("dot/preset.dot", DotPreorderDigraph(gr));
83
```
9.1.6 DotHighlightedDigraph

\[ \text{DotHighlightedDigraph}(\text{digraph}, \text{verts}[\text{, colour1, colour2}]) \]

Returns: A string.

DotHighlightedDigraph produces a graphical representation of the digraph \( \text{digraph} \), where the vertices in the list \( \text{verts} \), and edges between them, are drawn with colour \( \text{colour1} \) and all other vertices and edges in \( \text{digraph} \) are drawn with colour \( \text{colour2} \). If \( \text{colour1} \) and \( \text{colour2} \) are not given then DotHighlightedDigraph uses black and grey respectively.

Note that DotHighlightedDigraph does not validate the colours \( \text{colour1} \) and \( \text{colour2} \) - consult the GraphViz documentation to see what is available. See DotDigraph (9.1.2) for more details on the output.

Example

\[
\begin{align*}
\text{gap} & \text{> digraph := Digraph([[2, 3], [2], [1, 3]]);}
\text{<digraph with 3 vertices, 5 edges>}
\text{gap &> FileString(“dot/my_digraph.dot”,}
\text{ > DotHighlightedDigraph(digraph, [1, 2], “red”, “black”));
\end{align*}
\]

264

9.2 Reading and writing digraphs to a file

This section describes different ways to store and read graphs from a file in the Digraphs package.

Graph6

Graph6 is a graph data format for storing undirected graphs with no multiple edges nor loops of size up to \( 2^{36} - 1 \) in printable characters. The format consists of two parts. The first part uses one to eight bytes to store the number of vertices. And the second part is the upper half of the adjacency matrix converted into ASCII characters. For a more detail description see Graph6.

Sparse6

Sparse6 is a graph data format for storing undirected graphs with possibly multiple edges or loops. The maximal number of vertices allowed is \( 2^{36} - 1 \). The format consists of two parts. The first part uses one to eight bytes to store the number of vertices. And the second part only stores information about the edges. Therefore, the Sparse6 format return a more compact encoding then Graph6 for sparse graph, i.e. graphs where the number of edges is much less than the number of vertices squared. For a more detail description see Sparse6.

Digraph6

Digraph6 is a new format based on Graph6, but designed for digraphs. The entire adjacency matrix is stored, and therefore there is support for directed edges and single-vertex loops. However, multiple edges are not supported.

DiSparse6

DiSparse6 is a new format based on Sparse6, but designed for digraphs. In this format the list of edges is partitioned into increasing and decreasing edges, depending whether the edge has its source bigger than the range. Then both sets of edges are written separately in Sparse6 format with a separation symbol in between.
9.2.1 String

▷ String\( (\text{digraph}) \) (attribute)
▷ PrintString\( (\text{digraph}) \) (operation)

Returns: A string.

Returns a string string such that \( \text{EvalString} \( \text{string} \) \) is equal to \( \text{digraph} \), and has the same mutability. See \text{EvalString} (Reference: \text{EvalString}).

The methods installed for \text{String} make some attempts to ensure that string has as short a length as possible, but there may exist shorter strings that also evaluate to \( \text{digraph} \).

It is possible that \text{string} may contain escaped special characters. To obtain a representation of \( \text{digraph} \) that can be entered as GAP input, please use Print (Reference: Print). Note that Print for a digraph delegates to PrintString, which delegates to String.

```
 gap> D := CycleDigraph(3);
 <immutable cycle digraph with 3 vertices>
gap> Print(D);
CycleDigraph(3);
gap> G := PetersenGraph(IsMutableDigraph);
<mutable digraph with 10 vertices, 30 edges>
gap> String(G);
"DigraphFromGraph6String(IsMutableDigraph, "IheA@GUaO");"
gap> Print(last);
DigraphFromGraph6String(IsMutableDigraph, "IheA@GUaO");
gap> DigraphFromGraph6String(IsMutableDigraph, "IheA@GUaO");
<mutable digraph with 10 vertices, 30 edges>
```

9.2.2 DigraphFromGraph6String

▷ DigraphFromGraph6String(\([\text{filt,}]\text{str}\)) (operation)
▷ DigraphFromDigraph6String(\([\text{filt,}]\text{str}\)) (operation)
▷ DigraphFromSparse6String(\([\text{filt,}]\text{str}\)) (operation)
▷ DigraphFromDiSparse6String(\([\text{filt,}]\text{str}\)) (operation)

Returns: A digraph.

If \text{str} is a string encoding a graph in Graph6, Digraph6, Sparse6 or DiSparse6 format, then the corresponding function returns a digraph. In the case of either Graph6 or Sparse6, formats which do not support directed edges, this will be a digraph such that for every edge, the edge going in the opposite direction is also present.

Each of these functions takes an optional first argument \text{filt}, which should be either \text{IsMutableDigraph} (3.1.2) or \text{IsImmutableDigraph} (3.1.3), and which specifies whether the output digraph shall be mutable or immutable. If no first argument is provided, then an immutable digraph is returned by default.

```
 gap> DigraphFromGraph6String("?");
<immutable empty digraph with 0 vertices>
gap> DigraphFromGraph6String("C");
<immutable symmetric digraph with 4 vertices, 8 edges>
gap> DigraphFromGraph6String("H?AAEM{";
<immutable symmetric digraph with 9 vertices, 22 edges>
gap> DigraphFromGraph6String("&?");
<immutable empty digraph with 0 vertices>
```
Digraphs

gap> DigraphFromDigraph6String(IsMutableDigraph, "&DOOW?");
<mutable digraph with 5 vertices, 5 edges>

gap> DigraphFromDigraph6String("&CQFG");
<immutable digraph with 4 vertices, 6 edges>

gap> DigraphFromDigraph6String("&IM[SrKLc~1hesbU[F_"");
<immutable digraph with 10 vertices, 51 edges>

gap> DigraphFromDiSparse6String(".CaWBGA?b");
<immutable multidigraph with 4 vertices, 9 edges>

9.2.3 Graph6String

▷ Graph6String(digraph) (operation)
▷ Digraph6String(digraph) (operation)
▷ Sparse6String(digraph) (operation)
▷ DiSparse6String(digraph) (operation)

Returns: A string.

These four functions return a highly compressed string fully describing the digraph digraph. Graph6 and Digraph6 are formats best used on small, dense graphs, if applicable. For larger, sparse graphs use Sparse6 and Disparse6 (this latter also preserves multiple edges).

See WriteDigraphs (9.2.6).

Example

gap> gr := Digraph([ [2, 3], [1], [1] ]);  
<immutable digraph with 3 vertices, 4 edges>

gap> Sparse6String(gr);
".Bc"

gap> DiSparse6String(gr);
".Bc{f"

9.2.4 DigraphFile

▷ DigraphFile(filename[, coder][, mode]) (function)

Returns: An IO file object.

If filename is a string representing the name of a file, then DigraphFile returns an IO package file object for that file.

If the optional argument coder is specified and is a function which either encodes a digraph as a string, or decodes a string into a digraph, then this function will be used when reading or writing to the returned file object. If the optional argument coder is not specified, then the encoding of the digraphs in the returned file object must be specified in the the file extension. The file extension must be one of: .g6, .s6, .d6, .ds6, .txt, .p, or .pickle; more details of these file formats is given below.

If the optional argument mode is specified, then it must be one of: ",w" (for write), ",a" (for append), or ",r" (for read). If mode is not specified, then ",r" is used by default.

If filename ends in one of: .gz, .bz2, or .xz, then the digraphs which are read from, or written to, the returned file object are decompressed, or compressed, appropriately.

The file object returned by DigraphFile can be given as the first argument for either of the functions ReadDigraphs (9.2.5) or WriteDigraphs (9.2.6). The purpose of this is to reduce the overhead of recreating the file object inside the functions ReadDigraphs (9.2.5) or WriteDigraphs (9.2.6) when, for example, reading or writing many digraphs in a loop.

The currently supported file formats, and associated filename extensions, are:
graph6 (.g6)
A standard and widely-used format for undirected graphs, with no support for loops or multiple edges. Only symmetric graphs are allowed – each edge is combined with its converse edge to produce a single undirected edge. This format is best used for "dense" graphs – those with many edges per vertex.

sparse6 (.s6)
Unlike graph6, sparse6 has support for loops and multiple edges. However, its use is still limited to symmetric graphs. This format is better-suited to "sparse" graphs – those with few edges per vertex.

digraph6 (.d6)
This format is based on graph6, but stores direction information - therefore is not limited to symmetric graphs. Loops are allowed, but multiple edges are not. Best compression with "dense" graphs.

disparse6 (.ds6)
Any type of digraph can be encoded in disparse6: directions, loops, and multiple edges are all allowed. Similar to sparse6, this has the best compression rate with "sparse" graphs.

plain text (.txt)
This is a human-readable format which stores graphs in the form 0 7 0 8 1 7 2 8 3 8 4 8 5 8 6 8 i.e. pairs of vertices describing edges in a graph. More specifically, the vertices making up one edge must be separated by a single space, and pairs of vertices must be separated by two spaces.

See ReadPlainTextDigraph (9.2.13) for a more flexible way to store digraphs in a plain text file.

pickled (.p or .pickle)
Digraphs are pickled using the IO package. This is particularly good when the DigraphGroup (7.2.10) is non-trivial.

Example

```gap
gap> filename := Concatenation(DIGRAPHS_Dir(), "/tst/out/man.d6.gz");
gap> file := DigraphFile(filename, "w");
gap> for i in [1 .. 10] do
>   WriteDigraphs(file, Digraph([[1, 3], [2], [1, 2]]));
> od;
gap> IO_Close(file);
gap> file := DigraphFile(filename, "r");
gap> ReadDigraphs(file, 9);
<immutable digraph with 3 vertices, 5 edges>
gap> IO_Close(file);
```

9.2.5 ReadDigraphs

▷ ReadDigraphs(filename[, decoder][, n])

Returns: A digraph, or a list of digraphs.

If filename is a string containing the name of a file containing encoded digraphs or an IO file object created using DigraphFile (9.2.4), then ReadDigraphs returns the digraphs encoded in the
file as a list. Note that if filename is a compressed file, which has been compressed appropriately to
give a filename extension of .gz, .bz2, or .xz, then ReadDigraphs can read filename without it
first needing to be decompressed.

If the optional argument decoder is specified and is a function which decodes a string into a
digraph, then ReadDigraphs will use decoder to decode the digraphs contained in filename.

If the optional argument n is specified, then ReadDigraphs returns the n-th digraph encoded in the file filename.

If the optional argument decoder is not specified, then ReadDigraphs will deduce which decoder
to use based on the filename extension of filename (after removing the compression-related filename
extensions .gz, .bz2, and .xz). For example, if the filename extension is .g6, then ReadDigraphs
will use the graph6 decoder DigraphFromGraph6String (9.2.2).

The currently supported file formats, and associated filename extensions, are:

**graph6 (.g6)**
A standard and widely-used format for undirected graphs, with no support for loops or multiple
edges. Only symmetric graphs are allowed – each edge is combined with its converse edge to
produce a single undirected edge. This format is best used for "dense" graphs – those with many
edges per vertex.

**sparse6 (.s6)**
Unlike graph6, sparse6 has support for loops and multiple edges. However, its use is still limited
to symmetric graphs. This format is better-suited to "sparse" graphs – those with few edges per vertex.

**digraph6 (.d6)**
This format is based on graph6, but stores direction information - therefore is not limited to
symmetric graphs. Loops are allowed, but multiple edges are not. Best compression with
"dense" graphs.

**disparse6 (.ds6)**
Any type of digraph can be encoded in disparse6: directions, loops, and multiple edges are all
allowed. Similar to sparse6, this has the best compression rate with "sparse" graphs.

**plain text (.txt)**
This is a human-readable format which stores graphs in the form
0 7 0 8 1 7 2 8 3 8 4 8 5 8 6 8 i.e. pairs of vertices describing edges in a graph. More specifically, the vertices
making up one edge must be separated by a single space, and pairs of vertices must be separated
by two spaces.

See ReadPlainTextDigraph (9.2.13) for a more flexible way to store digraphs in a plain text
file.

**pickled (.p or .pickle)**
Digraphs are pickled using the IO package. This is particularly good when the DigraphGroup
(7.2.10) is non-trivial.

---

Example

```gap
gap> ReadDigraphs(  
> Concatenation(DIGRAPHS_Dir(), "/data/graph5.g6.gz"), 10);  
<immutable symmetric digraph with 5 vertices, 8 edges>
```

```gap
gap> ReadDigraphs(  
> Concatenation(DIGRAPHS_Dir(), "/data/graph5.g6.gz"), 10);  
<immutable symmetric digraph with 5 vertices, 8 edges>
```
 Digraphs

> Concatenation(DIGRAPHS_Dir(), "/data/graph5.g6.gz"), 17);
> <immutable symmetric digraph with 5 vertices, 12 edges>
gap> ReadDigraphs(
> Concatenation(DIGRAPHS_Dir(), "/data/tree9.4.txt"));
[ <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
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 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>,
 <immutable digraph with 9 vertices, 8 edges>
 ]

9.2.6 WriteDigraphs

▷ WriteDigraphs(filename, digraphs[, encoder][, mode]) (function)

If digraphs is a list of digraphs or a digraph and filename is a string or an IO file object created using DigraphFile (9.2.4), then WriteDigraphs writes the digraphs to the file represented by filename. If the supplied filename ends in one of the extensions .gz, .bz2, or .xz, then the file will be compressed appropriately. Excluding these extensions, if the file ends with an extension in the list below, the corresponding graph format will be used to encode it. If such an extension is not included, an appropriate format will be chosen intelligently, and an extension appended, to minimise file size.

For more verbose information on the progress of the function, set the info level of InfoDigraphs to 1 or higher, using SetInfoLevel.

The currently supported file formats are:

graph6 (.g6)
A standard and widely-used format for undirected graphs, with no support for loops or multiple edges. Only symmetric graphs are allowed – each edge is combined with its converse edge to produce a single undirected edge. This format is best used for "dense" graphs – those with many edges per vertex.

sparse6 (.s6)
Unlike graph6, sparse6 has support for loops and multiple edges. However, its use is still limited to symmetric graphs. This format is better-suited to "sparse" graphs – those with few edges per vertex.

digraph6 (.d6)
This format is based on graph6, but stores direction information - therefore is not limited to symmetric graphs. Loops are allowed, but multiple edges are not. Best compression with "dense" graphs.
Any type of digraph can be encoded in disparse6: directions, loops, and multiple edges are all allowed. Similar to sparse6, this has the best compression rate with "sparse" graphs.

**Plain text (.txt)**

This is a human-readable format which stores graphs in the form 0 7 0 8 1 7 2 8 3 8 4 8 5 8 6 8 i.e. pairs of vertices describing edges in a graph. More specifically, the vertices making up one edge must be separated by a single space, and pairs of vertices must be separated by two spaces.

See ReadPlainTextDigraph (9.2.13) for a more flexible way to store digraphs in a plain text file.

**Pickled (.p or .pickle)**

Digraphs are pickled using the IO package. This is particularly good when the DigraphGroup (7.2.10) is non-trivial.

```
gap> grs := [];;
gap> grs[1] := Digraph([]);
<immutable empty digraph with 0 vertices>
gap> grs[2] := Digraph([[1, 3], [2], [1, 2]]);
<immutable digraph with 3 vertices, 5 edges>
gap> grs[3] := Digraph([>
  [6, 7], [6, 9], [1, 3, 4, 5, 8, 9],
  [1, 2, 3, 4, 5, 6, 7, 10], [1, 5, 6, 7, 10], [2, 4, 5, 9, 10],
  [3, 4, 5, 6, 7, 8, 9, 10], [1, 3, 5, 7, 8, 9], [1, 2, 5],
  [1, 2, 4, 6, 7, 8]]);
<immutable digraph with 10 vertices, 51 edges>
gap> filename := Concatenation(DIGRAPHS_Dir(), "/tst/out/man.d6.gz");
gap> WriteDigraphs(filename, grs, "w");
IO_OK
gap> ReadDigraphs(filename);
[ <immutable empty digraph with 0 vertices>,
  <immutable digraph with 3 vertices, 5 edges>,
  <immutable digraph with 10 vertices, 51 edges> ]
```

### 9.2.7 IteratorFromDigraphFile

**IteratorFromDigraphFile**

- **IteratorFromDigraphFile** *(filename[, decoder])*  (function)
  
  **Returns**: An iterator.

  If *filename* is a string representing the name of a file containing encoded digraphs, then IteratorFromDigraphFile returns an iterator for which the value of NextIterator (Reference: NextIterator) is the next digraph encoded in the file.

  If the optional argument *decoder* is specified and is a function which decodes a string into a digraph, then IteratorFromDigraphFile will use *decoder* to decode the digraphs contained in *filename*.

  The purpose of this function is to easily allow looping over digraphs encoded in a file when loading all of the encoded digraphs would require too much memory.

  To see what file types are available, see WriteDigraphs (9.2.6).
Example

```gap
gap> filename := Concatenation(DIGRAPHS_Dir(), "/tst/out/man.d6.gz");;
gap> file := DigraphFile(filename, "w");;
gap> for i in [1 .. 10] do
  > WriteDigraphs(file, Digraph([[1, 3], [2], [1, 2]]));
  > od;
gap> IO_Close(file);

gap> iter := IteratorFromDigraphFile(filename);
<iterator>
gap> for x in iter do od;
```

9.2.8 DigraphPlainTextLineEncoder

- **DigraphPlainTextLineEncoder**(delimiter1[, delimiter2], offset)
- **DigraphPlainTextLineDecoder**(delimiter1[, delimiter2], offset)

**Returns:** A string.

These two functions return a function which encodes or decodes a digraph in a plain text format. `DigraphPlainTextLineEncoder` returns a function which takes a single digraph as an argument. The function returns a string describing the edges of that digraph; each edge is written as a pair of integers separated by the string `delimiter2`, and the edges themselves are separated by the string `delimiter1`. `DigraphPlainTextLineDecoder` returns the corresponding decoder function, which takes a string argument in this format and returns a digraph.

If only one delimiter is passed as an argument to `DigraphPlainTextLineDecoder`, it will return a function which decodes a single edge, returning its contents as a list of integers.

The argument `offset` should be an integer, which will describe a number to be added to each vertex before it is encoded, or after it is decoded. This may be used, for example, to label vertices starting at 0 instead of 1.

Note that the number of vertices of a digraph is not stored, and so vertices which are not connected to any edge may be lost.

Example

```gap
gap> gr := Digraph([[2, 3], [1], [1]]);
<immutable digraph with 3 vertices, 4 edges>
gap> enc := DigraphPlainTextLineEncoder(" ", " ", -1);
gap> dec := DigraphPlainTextLineDecoder(" ", " ", 1);
gap> enc(gr);
"0 1 0 2 1 0 2 0"
gap> dec(last);
<immutable digraph with 3 vertices, 4 edges>
```

9.2.9 TournamentLineDecoder

- **TournamentLineDecoder**(str)

**Returns:** A digraph.

This function takes a string `str`, decodes it, and then returns the tournament [see IsTournament (6.2.15)] which it defines, according to the following rules.

The characters of the string `str` represent the entries in the upper triangle of a tournament’s adjacency matrix. The number of vertices `n` will be detected from the length of the string and will be as large as possible.
The first character represents the possible edge 1 -> 2, the second represents 1 -> 3 and so on until 1 -> n; then the following character represents 2 -> 3, and so on up to the character which represents the edge n-1 -> n.

If a character of the string with corresponding edge i -> j is equal to 1, then the edge i -> j is present in the tournament. Otherwise, the edge i -> j is present instead. In this way, all the possible edges are encoded one-by-one.

**Example**

```gap
gap> gr := TournamentLineDecoder("100001");
<immutable digraph with 4 vertices, 6 edges>
gap> OutNeighbours(gr);
[ [ 2 ], [ ], [ 1, 2, 4 ], [ 1, 2 ] ]
```

9.2.10 **AdjacencyMatrixUpperTriangleLineDecoder**

▷ **AdjacencyMatrixUpperTriangleLineDecoder**(str)  (operation)

**Returns:** A digraph.

This function takes a string str, decodes it, and then returns the topologically sorted digraph [see DigraphTopologicalSort (5.1.8)] which it defines, according to the following rules.

The characters of the string str represent the entries in the upper triangle of a digraph’s adjacency matrix. The number of vertices n will be detected from the length of the string and will be as large as possible.

The first character represents the possible edge 1 -> 2, the second represents 1 -> 3 and so on until 1 -> n; then the following character represents 2 -> 3, and so on up to the character which represents the edge n-1 -> n. If a character of the string with corresponding edge i -> j is equal to 1, then this edge is present in the digraph. Otherwise, it is not present. In this way, all the possible edges are encoded one-by-one.

In particular, note that there exists no edge [i, j] if j ≤ i. In other words, the digraph will be topologically sorted.

**Example**

```gap
gap> gr := AdjacencyMatrixUpperTriangleLineDecoder("100001");
<immutable digraph with 4 vertices, 2 edges>
gap> OutNeighbours(gr);
[ [ 2 ], [ ], [ 4 ], [ ] ]
gap> gr := AdjacencyMatrixUpperTriangleLineDecoder("111111x111");
<immutable digraph with 5 vertices, 9 edges>
gap> OutNeighbours(gr);
[ [ 2, 3, 4, 5 ], [ 3, 4 ], [ 4, 5 ], [ 5 ], [ ] ]
```

9.2.11 **TCodeDecoder**

▷ **TCodeDecoder**(str)  (operation)

**Returns:** A digraph.

If str is a string consisting of at least two non-negative integers separated by spaces, then this function will attempt to return the digraph which it defines as a TCode string.

The first integer of the string defines the number of vertices v in the digraph, and the second defines the number of edges e. The following 2e integers should be vertex numbers in the range [0 .. v-1]. These integers are read in pairs and define the digraph’s edges. This function will return an error if str has fewer than 2e+2 entries.
Note that the vertex numbers will be incremented by 1 in the digraph returned. Hence the string fragment \(0 \ 6\) will describe the edge \([1,7]\).

**Example**

```gap
gap> gr := TCodeDecoder("3 2 0 2 2 1");
<immutable digraph with 3 vertices, 2 edges>
gap> OutNeighbours(gr);
[ [ 3 ], [ ], [ 2 ] ]
gap> gr := TCodeDecoder("12 3 0 10 5 2 8 8");
<immutable digraph with 12 vertices, 3 edges>
gap> OutNeighbours(gr);
[ [ 11 ], [ ], [ ], [ ], [ ], [ 3 ], [ ], [ ], [ 9 ], [ ],
  [ ], [ ] ]
```

### 9.2.12 PlainTextString

- PlainTextString(digraph) *(operation)*
- DigraphFromPlainTextString(s) *(operation)*

**Returns:** A string.

*PlainTextString* takes a single digraph, and returns a string describing the edges of that digraph. *DigraphFromPlainTextString* takes such a string and returns the digraph which it describes. Each edge is written as a pair of integers separated by a single space. The edges themselves are separated by a double space. Vertex numbers are reduced by 1 when they are encoded, so that vertices in the string are labelled starting at 0.

Note that the number of vertices of a digraph is not stored, and so vertices which are not connected to any edge may be lost.

The operation *DigraphFromPlainTextString* takes an optional first argument IsMutableDigraph (3.1.2) or IsImmutableDigraph (3.1.3), which specifies whether the output digraph shall be mutable or immutable. If no first argument is provided, then an immutable digraph is returned by default.

**Example**

```gap
gap> gr := Digraph([[2, 3], [1], [1]]);
<immutable digraph with 3 vertices, 4 edges>
gap> PlainTextString(gr);
"0 1 0 2 1 0 2 0"
gap> DigraphFromPlainTextString(last);
<immutable digraph with 3 vertices, 4 edges>
```

### 9.2.13 WritePlainTextDigraph

- WritePlainTextDigraph(filename, digraph, delimiter, offset) *(function)*
- ReadPlainTextDigraph(filename, delimiter, offset, ignore) *(operation)*

These functions write and read a single digraph in a human-readable plain text format as follows: each line contains a single edge, and each edge is written as a pair of integers separated by the string *delimiter*.

*filename* should be the name of a file which will be written to or read from, and *offset* should be an integer which is added to each vertex number as it is written or read. For example, if *WritePlainTextDigraph* is called with *offset* = -1, then the vertices will be numbered in the file
starting from 0 instead of 1 - ReadPlainTextDigraph would then need to be called with offset 1 to convert back to the original graph.

`ignore` should be a list of characters which will be ignored when reading the graph.

Example

```
gap> gr := Digraph([[1, 2, 3], [1, 1], [2]]);
<immutable multidigraph with 3 vertices, 6 edges>
gap> filename := Concatenation(DIGRAPHS_Dir(), "/tst/out/plain.txt");
gap> WritePlainTextDigraph(filename, gr, ",", -1);
gap> ReadPlainTextDigraph(filename, ",", 1, ['/', '%']);
<immutable multidigraph with 3 vertices, 6 edges>
```

### 9.2.14 WriteDIMACSDigraph

- **WriteDIMACSDigraph**(filename, digraph)  
  (operation)

- **ReadDIMACSDigraph**(filename)  
  (operation)

These operations write or read the single symmetric digraph `digraph` to or from a file in DIMACS format, as appropriate. The operation `WriteDIMACSDigraph` records the vertices and edges of `digraph`. The vertex labels of `digraph` will be recorded only if they are integers. See `IsSymmetricDigraph` (6.2.14) and `DigraphVertexLabels` (5.1.10).

The first argument `filename` should be the name of the file which will be written to or read from. A file can contain one symmetric digraph in DIMACS format. If `filename` ends in one of .gz, .bz2, or .xz, then the file is compressed, or decompressed, appropriately.

The DIMACS format is described as follows. Each line in the DIMACS file has one of four types:

- A line beginning with `c` and followed by any number of characters is a comment line, and is ignored.

- A line beginning with `p` defines the numbers of vertices and edges the digraph. This line has the format `p edge <nr_vertices> <nr_edges>`, where `<nr_vertices>` and `<nr_edges>` are replaced by the relevant integers. There must be exactly one such line in the file, and it must occur before any of the following kinds of line.

  Although it is required to be present, the value of `<nr_edges>` will be ignored. The correct number of edges will be deduced from the rest of the information in the file.

- A line of the form `e <v> <w>`, where `<v>` and `<w>` are integers in the range `[1 .. <nr_vertices>]`, specifies that there is a (symmetric) edge in the digraph between the vertices `<v>` and `<w>`. A symmetric edge only needs to be defined once; an additional line `e <v> <w>`, or `e <w> <v>`, will be interpreted as an additional, multiple, edge. Loops are permitted.

- A line of the form `n <v> <label>`, where `<v>` is an integer in the range `[1 .. <nr_vertices>]` and `<label>` is an integer, signifies that the vertex `<v>` has the label `<label>` in the digraph. If a label is not specified for a vertex, then `ReadDIMACSDigraph` will assign the label 1, according to the DIMACS specification.

A detailed definition of the DIMACS format can be found at [http://mat.gsia.cmu.edu/COLOR/general/ccformat.ps](http://mat.gsia.cmu.edu/COLOR/general/ccformat.ps), in Section 2.1. Note that optional descriptor lines, as described in Section 2.1, will be ignored.
Example

```gap
gap> gr := Digraph([[2], [1, 3, 4], [2, 4], [2, 3]]);
<immutable digraph with 4 vertices, 8 edges>
gap> filename := Concatenation(DIGRAPHS_Dir(),
"/tst/out/dimacs.dimacs");
gap> WriteDIMACSDigraph(filename, gr);
gap> ReadDIMACSDigraph(filename);
<immutable digraph with 4 vertices, 8 edges>
```
Appendix A

Grape to Digraphs Command Map

Below is a table of Grape commands with the Digraphs counterparts. The sections in this chapter correspond to the chapters in the Grape manual.

A.1 Functions to construct and modify graphs

*The table in this section contains more information when viewed in html format.*

<table>
<thead>
<tr>
<th>Grape command</th>
<th>Digraphs command</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph</td>
<td>Digraph (3.1.7)</td>
</tr>
<tr>
<td>EdgeOrbitsGraph</td>
<td>EdgeOrbitsDigraph (3.1.10)</td>
</tr>
<tr>
<td>NullGraph</td>
<td>NullDigraph (3.5.16)</td>
</tr>
<tr>
<td>CompleteGraph</td>
<td>CompleteDigraph (3.5.11)</td>
</tr>
<tr>
<td>JohnsonGraph</td>
<td>JohnsonDigraph (3.5.23)</td>
</tr>
<tr>
<td>CayleyGraph</td>
<td>CayleyDigraph (3.1.12)</td>
</tr>
<tr>
<td>AddEdgeOrbit</td>
<td>DigraphAddEdgeOrbit (3.3.18)</td>
</tr>
<tr>
<td>RemoveEdgeOrbit</td>
<td>DigraphRemoveEdgeOrbit (3.3.23)</td>
</tr>
<tr>
<td>AssignVertexNames</td>
<td>SetDigraphVertexLabels (5.1.10)</td>
</tr>
</tbody>
</table>

A.2 Functions to inspect graphs, vertices and edges

*The table in this section contains more information when viewed in html format.*
### Functions to determine regularity properties of graphs

*The table in this section contains more information when viewed in html format.*

<table>
<thead>
<tr>
<th>Grape command</th>
<th>Digraphs command</th>
</tr>
</thead>
<tbody>
<tr>
<td>IsRegularGraph</td>
<td>IsOutRegularDigraph (6.5.2)</td>
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<tr>
<td>LocalParameters</td>
<td>None</td>
</tr>
<tr>
<td>GlobalParameters</td>
<td>None</td>
</tr>
<tr>
<td>IsDistanceRegular</td>
<td>IsDistanceRegularDigraph (6.5.4)</td>
</tr>
<tr>
<td>CollapsedAdjacencyMat</td>
<td>None</td>
</tr>
<tr>
<td>OrbitalGraphColadjMats</td>
<td>None</td>
</tr>
<tr>
<td>VertexTransitiveDRGs</td>
<td>None</td>
</tr>
</tbody>
</table>

### Some special vertex subsets of a graph

*The table in this section contains more information when viewed in html format.*

<table>
<thead>
<tr>
<th>Grape command</th>
<th>Digraphs command</th>
</tr>
</thead>
<tbody>
<tr>
<td>ConnectedComponent</td>
<td>DigraphConnectedComponent (5.4.10)</td>
</tr>
<tr>
<td>ConnectedComponents</td>
<td>DigraphConnectedComponents (5.4.9)</td>
</tr>
<tr>
<td>Bicomponents</td>
<td>DigraphBicomponents (5.4.13)</td>
</tr>
<tr>
<td>DistanceSet</td>
<td>DigraphDistanceSet (5.4.13)</td>
</tr>
<tr>
<td>Layers</td>
<td>DigraphLayers (5.4.32)</td>
</tr>
<tr>
<td>IndependentSet</td>
<td>DigraphIndependentSet (8.2.2)</td>
</tr>
</tbody>
</table>
### A.5 Functions to construct new graphs from old

*The table in this section contains more information when viewed in html format.*

<table>
<thead>
<tr>
<th>Grape command</th>
<th>Digraphs command</th>
</tr>
</thead>
<tbody>
<tr>
<td>InducedSubgraph</td>
<td>InducedSubdigraph (3.3.3)</td>
</tr>
<tr>
<td>DistanceSetInduced</td>
<td>None</td>
</tr>
<tr>
<td>DistanceGraph</td>
<td>DistanceDigraph (3.3.45)</td>
</tr>
<tr>
<td>ComplementGraph</td>
<td>DigraphDual (3.3.11)</td>
</tr>
<tr>
<td>PointGraph</td>
<td>None</td>
</tr>
<tr>
<td>EdgeGraph</td>
<td>EdgeUndirectedDigraph (3.3.41)</td>
</tr>
<tr>
<td>SwitchedGraph</td>
<td>None</td>
</tr>
<tr>
<td>UnderlyingGraph</td>
<td>DigraphSymmetricClosure (3.3.12)</td>
</tr>
<tr>
<td>QuotientGraph</td>
<td>QuotientDigraph (3.3.9)</td>
</tr>
<tr>
<td>BipartiteDouble</td>
<td>BipartiteDoubleDigraph (3.3.43)</td>
</tr>
<tr>
<td>GeodesicsGraph</td>
<td>None</td>
</tr>
<tr>
<td>CollapsedIndependentOrbitsGraph</td>
<td>None</td>
</tr>
<tr>
<td>CollapsedCompleteOrbitsGraph</td>
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</tr>
<tr>
<td>NewGroupGraph</td>
<td>None</td>
</tr>
</tbody>
</table>

### A.6 Vertex-Colouring and Complete Subgraphs

*The table in this section contains more information when viewed in html format.*

<table>
<thead>
<tr>
<th>Grape command</th>
<th>Digraphs command</th>
</tr>
</thead>
<tbody>
<tr>
<td>VertexColouring</td>
<td>DigraphGreedyColouring (7.3.15)</td>
</tr>
<tr>
<td>CompleteSubgraphs</td>
<td>DigraphCliques (8.1.4)</td>
</tr>
<tr>
<td>CompleteSubgraphsOfGivenSize</td>
<td>DigraphCliques (8.1.4)</td>
</tr>
</tbody>
</table>

### A.7 Automorphism groups and isomorphism testing for graphs

*The table in this section contains more information when viewed in html format.*

<table>
<thead>
<tr>
<th>Grape command</th>
<th>Digraphs command</th>
</tr>
</thead>
<tbody>
<tr>
<td>AutGroupGraph</td>
<td>AutomorphismGroup (7.2.2)</td>
</tr>
<tr>
<td>GraphIsomorphism</td>
<td>IsomorphismDigraphs (7.2.17)</td>
</tr>
<tr>
<td>IsIsomorphicGraph</td>
<td>IsIsomorphicDigraph (7.2.15)</td>
</tr>
<tr>
<td>GraphIsomorphismClassRepresentatives</td>
<td>None</td>
</tr>
</tbody>
</table>
References


<table>
<thead>
<tr>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; (for digraphs), 79</td>
</tr>
<tr>
<td>= (for digraphs), 79</td>
</tr>
<tr>
<td>AdjacencyMatrix, 91</td>
</tr>
<tr>
<td>AdjacencyMatrixMutableCopy, 91</td>
</tr>
<tr>
<td>AdjacencyMatrixUpperTriangleLine-Decoder, 217</td>
</tr>
<tr>
<td>AndrasfaiGraph, 52</td>
</tr>
<tr>
<td>ArticulationPoints, 109</td>
</tr>
<tr>
<td>AsBinaryRelation, 19</td>
</tr>
<tr>
<td>AsDigraph</td>
</tr>
<tr>
<td>for a binary relation, 19</td>
</tr>
<tr>
<td>AsGraph, 21</td>
</tr>
<tr>
<td>AsMonoid, 124</td>
</tr>
<tr>
<td>AsSemigroup</td>
</tr>
<tr>
<td>for a filter and a digraph, 124</td>
</tr>
<tr>
<td>for a filter, semilattice digraph, and two lists, 125</td>
</tr>
<tr>
<td>AsTransformation, 21</td>
</tr>
<tr>
<td>AutomorphismGroup</td>
</tr>
<tr>
<td>for a digraph and a homogeneous list, 161</td>
</tr>
<tr>
<td>for a digraph, homogeneous list, and list, 162</td>
</tr>
<tr>
<td>BananaTree, 53</td>
</tr>
<tr>
<td>BinaryTree, 53</td>
</tr>
<tr>
<td>BinomialTreeGraph, 54</td>
</tr>
<tr>
<td>BipartiteDoubleDigraph, 48</td>
</tr>
<tr>
<td>BishopGraph, 54</td>
</tr>
<tr>
<td>BishopGraph, 54</td>
</tr>
<tr>
<td>BlissAutomorphismGroup</td>
</tr>
<tr>
<td>for a digraph, 160</td>
</tr>
<tr>
<td>for a digraph and homogeneous list, 160</td>
</tr>
<tr>
<td>for a digraph, homogeneous list, and list, 160</td>
</tr>
<tr>
<td>BlissCanonicalDigraph, 165</td>
</tr>
<tr>
<td>BlissCanonicalLabelling</td>
</tr>
<tr>
<td>for a digraph, 163</td>
</tr>
<tr>
<td>for a digraph and a list, 164</td>
</tr>
<tr>
<td>BondyGraph, 55</td>
</tr>
<tr>
<td>BookGraph, 56</td>
</tr>
<tr>
<td>BooleanAdjacencyMatrix, 93</td>
</tr>
<tr>
<td>BooleanAdjacencyMatrixMutableCopy, 93</td>
</tr>
<tr>
<td>Bridges, 110</td>
</tr>
<tr>
<td>CayleyDigraph, 18</td>
</tr>
<tr>
<td>ChainDigraph, 57</td>
</tr>
<tr>
<td>CharacteristicPolynomial, 92</td>
</tr>
<tr>
<td>ChromaticNumber, 185</td>
</tr>
<tr>
<td>CirculantGraph, 57</td>
</tr>
<tr>
<td>CliqueNumber, 195</td>
</tr>
<tr>
<td>CliquesFinder, 191</td>
</tr>
<tr>
<td>CompleteBipartiteDigraph, 58</td>
</tr>
<tr>
<td>CompleteDigraph, 58</td>
</tr>
<tr>
<td>CompleteMultipartiteDigraph, 59</td>
</tr>
<tr>
<td>ConormalProduct, 43</td>
</tr>
<tr>
<td>CycleDigraph, 59</td>
</tr>
<tr>
<td>CycleGraph, 60</td>
</tr>
<tr>
<td>DegreeMatrix, 99</td>
</tr>
<tr>
<td>Digraph, 13</td>
</tr>
<tr>
<td>for a group, list, function, and function, 13</td>
</tr>
<tr>
<td>for a list and function, 13</td>
</tr>
<tr>
<td>Digraph6String, 211</td>
</tr>
<tr>
<td>DigraphAbsorptionExpectedSteps, 117</td>
</tr>
<tr>
<td>DigraphAbsorptionProbabilities, 116</td>
</tr>
<tr>
<td>DigraphAddAllLoops, 48</td>
</tr>
<tr>
<td>DigraphAddAllLoopsAttr, 48</td>
</tr>
<tr>
<td>DigraphAddEdge</td>
</tr>
<tr>
<td>for a digraph and an edge, 33</td>
</tr>
<tr>
<td>for a digraph, source, and range, 33</td>
</tr>
<tr>
<td>DigraphAddEdgeOrbit, 34</td>
</tr>
<tr>
<td>DigraphAddEdges, 34</td>
</tr>
<tr>
<td>DigraphAddVertex, 32</td>
</tr>
<tr>
<td>DigraphAddVertices</td>
</tr>
<tr>
<td>for a digraph and a list of labels, 32</td>
</tr>
<tr>
<td>for a digraph and an integer, 32</td>
</tr>
<tr>
<td>DigraphAdjacencyFunction, 93</td>
</tr>
<tr>
<td>DigraphAllSimpleCircuits, 119</td>
</tr>
</tbody>
</table>
DigraphBicomponents, 109
DigraphByAdjacencyMatrix, 15
DigraphByEdges, 16
DigraphByInNeighbors, 17
DigraphByInNeighbours, 17
DigraphByOutNeighboursType, 13
DigraphCartesianProduct
  for a list of digraphs, 42
  for a positive number of digraphs, 42
DigraphCartesianProductProjections, 46
DigraphClique, 193
DigraphCliques, 194
DigraphCliquesReps, 194
DigraphClosure, 49
DigraphColouring
  for a digraph and a number of colours, 184
DigraphConnectedComponent, 107
DigraphConnectedComponents, 106
DigraphCopy, 22
DigraphCopySameMutability, 22
DigraphCore, 186
DigraphCycle, 59
DigraphDegeneracy, 120
DigraphDegeneracyOrdering, 121
DigraphDiameter, 102
DigraphDijkstra
  for a source, 123
  for a source and target, 123
DigraphDirectProduct
  for a list of digraphs, 42
  for a positive number of digraphs, 42
DigraphDirectProductProjections, 46
DigraphDisjointUnion
  for a list of digraphs, 40
  for an arbitrary number of digraphs, 40
DigraphDistanceSet
  for a digraph, a pos int, and a list, 104
  for a digraph, a pos int, and an int, 104
DigraphDual, 29
DigraphDualAttr, 29
DigraphEdgeLabel, 87
DigraphEdgeLabels, 88
DigraphEdges, 83
DigraphEdgeUnion
  for a list of digraphs, 40
  for a positive number of digraphs, 40
DigraphEmbedding, 180
DigraphEpiMorphism, 179
DigraphFamily, 13
DigraphFile, 211
DigraphFloydWarshall, 112
DigraphFromDigraph6String, 210
DigraphFromDiSparse6String, 210
DigraphFromGraph6String, 210
DigraphFromPlainTextString, 218
DigraphFromSparse6String, 210
DigraphGirth, 104
DigraphGreedyColouring
  for a digraph, 185
  for a digraph and vertex order, 185
  for a digraph and vertex order function, 185
DigraphGroup, 166
DigraphHasAVertex, 131
DigraphHasLoops, 132
DigraphHasNoVertices, 131
DigraphHomomorphism, 178
DigraphImmutableCopy, 22
DigraphImmutableCopyIfImmutable, 22
DigraphImmutableCopyIfMutable, 22
DigraphIndependentSet, 196
DigraphIndependentSets, 198
DigraphIndependentSetsReps, 198
DigraphInEdges, 89
DigraphJoin
  for a list of digraphs, 41
  for a positive number of digraphs, 41
DigraphJoinTable, 143
DigraphLayers, 120
DigraphLongestDistanceFromVertex, 103
DigraphLongestSimpleCircuit, 120
DigraphLoops, 99
DigraphMaximalClique, 193
DigraphMaximalCliques, 194
DigraphMaximalCliquesAttr, 194
DigraphMaximalCliquesReps, 194
DigraphMaximalCliquesRepsAttr, 194
DigraphMaximalIndependentSet, 196
DigraphMaximalIndependentSets, 198
DigraphMaximalIndependentSetsAttr, 198
DigraphMaximalIndependentSetsReps, 198
DigraphMaximalIndependentSetsRepsAttr, 198
DigraphMaximalMatching, 90
DigraphMaximumMatching, 91
DigraphMeetTable, 143
DigraphMonomorphism, 178
DigraphMutableCopy, 22
DigraphMutableCopyIfImmutable, 22
DigraphMutableCopyIfMutable, 22
DigraphMycielskian, 49
DigraphMycielskianAttr, 49
DigraphNrConnectedComponents, 106
DigraphNrEdges, 83
DigraphNrLoops, 84
DigraphNrStronglyConnectedComponents, 107
DigraphNrVertices, 82
DigraphOddGirth, 105
DigraphOrbitReps, 167
DigraphOrbits, 167
DigraphOutEdges, 89
DigraphPath, 115
DigraphPeriod, 111
DigraphPlainTextLineDecoder, 216
DigraphPlainTextLineEncoder, 216
DigraphRandomWalk, 116
DigraphRange, 94
DigraphReflexiveTransitiveClosure, 30
DigraphReflexiveTransitiveClosureAttr, 30
DigraphReflexiveTransitiveReduction, 31
DigraphReflexiveTransitiveReductionAttr, 31
DigraphRemoveAllMultipleEdges, 39
DigraphRemoveAllMultipleEdgesAttr, 39
DigraphRemoveEdge
  for a digraph and an edge, 36
  for a digraph, source, and range, 36
  for a digraph, source, and range, 36
DigraphRemoveEdgeOrbit, 37
DigraphRemoveEdges, 38
DigraphRemoveLoops, 38
DigraphRemoveLoopsAttr, 38
DigraphRemoveVertex, 35
DigraphRemoveVertices, 36
DigraphReverse, 28
DigraphReverseAttr, 28
DigraphReverseEdge
  for a digraph and an edge, 39
  for a digraph, source, and range, 39
DigraphReverseEdges
  for a digraph and a list of edges, 39
Digraphs package overview, 5
DigraphSchreierVector, 168
DigraphShortestDistance
  for a digraph and a list, 102
  for a digraph and two vertices, 102
  for a digraph, a list, and a list, 102
DigraphShortestDistances, 103
DigraphShortestPath, 115
DigraphShortestPathSpanningTree, 27
DigraphSinks, 84
DigraphsMakeDoc, 10
DigraphSource, 94
DigraphSources, 85
DigraphsRespectsColouring, 183
DigraphStabilizer, 168
DigraphsTestExtreme, 11
DigraphsTestInstall, 11
DigraphsTestStandard, 11
DigraphStronglyConnectedComponent, 108
DigraphStronglyConnectedComponents, 107
DigraphsUseBliss, 159
DigraphsUseNauty, 159
DigraphSymmetricClosure, 29
DigraphSymmetricClosureAttr, 29
DigraphTopologicalSort, 85
DigraphTransitiveClosure, 30
DigraphTransitiveClosureAttr, 30
DigraphTransitiveReduction, 31
DigraphTransitiveReductionAttr, 31
DigraphUndirectedGirth, 106
DigraphVertexLabel, 85
DigraphVertexLabels, 86
DigraphVertices, 82
DigraphWelshPowellOrder, 185
DiSparse6String, 211
DistanceDigraph
  for digraph and int, 49
  for digraph and list, 49
Dominators, 117
DominatorTree, 118
DotColoredDigraph, 201
DotDigraph, 201
DotEdgeColoredDigraph, 201
Digraphs

IsDigraphIsomorphism, 173
for digraphs and transformation or permutation, 173
IsDigraphMonomorphism, 181
for digraphs and a permutation or transformation, 181
IsDigraphPath, 113
for a digraph and list, 113
IsDigraphWithAdjacencyFunction, 13
IsDirectedTree, 150
IsDistanceRegularDigraph, 145
IsDistributiveLatticeDigraph, 144
IsEdgeTransitive, 155
IsEmptyDigraph, 135
IsEquivalenceDigraph, 135
IsEulerianDigraph, 152
IsFunctionalDigraph, 136
IsHamiltonianDigraph, 152
IsImmutableDigraph, 12
IsIndependentSet, 196
IsInRegularDigraph, 144
IsIsomorphicDigraph
for digraphs, 169
for digraphs and homogeneous lists, 169
IsJoinSemilatticeDigraph, 142
IsLatticeDigraph, 142
IsLatticeEmbedding
for digraphs and a permutation or transformation, 187
IsLatticeEndomorphism
for digraphs and a permutation or transformation, 187
IsLatticeEpimorphism
for digraphs and a permutation or transformation, 187
IsLatticeHomomorphism
for digraphs and a permutation or transformation, 187
IsLatticeMonomorphism
for digraphs and a permutation or transformation, 187
IsLowerSemimodularDigraph, 143
IsMatching, 90
IsMaximalClue, 190
IsMaximalIndependentSet, 196
IsMaximalMatching, 90
IsMaximumMatching, 90
IsMeetSemilatticeDigraph, 142
IsMultiDigraph, 137
IsMutableDigraph, 12
IsNonemptyDigraph, 137
IsNullDigraph, 135
IsomorphismDigraphs
for digraphs, 170
for digraphs and homogeneous lists, 171
IsOuterPlanarDigraph, 154
IsOutRegularDigraph, 145
IsPartialOrderDigraph, 141
IsPerfectMatching, 90
IsPermutationDigraph, 136
IsPlanarDigraph, 154
IsPreorderDigraph, 141
IsQuasiorderDigraph, 141
IsReachable, 113
IsReflexiveDigraph, 138
IsRegularDigraph, 145
IsStronglyConnectedDigraph, 149
IsSubdigraph, 79
IsSymmetricDigraph, 138
IsTournament, 139
IsTransitiveDigraph, 139
IsUndirectedForest, 151
IsUndirectedSpanningForest, 80
IsUndirectedSpanningTree, 80
IsUndirectedTree, 151
IsUpperSemimodularDigraph, 143
IsVertexTransitive, 156
IteratorFromDigraphFile, 215
IteratorOfPaths, 118
JohnsonDigraph, 63
KellerGraph, 64
KingsGraph, 64
KneserGraph, 65
KnightsGraph, 66
KuratowskiOuterPlanarSubdigraph, 126
KuratowskiPlanarSubdigraph, 126
LaplacianMatrix, 100
LatticeDigraphEmbedding, 187
LexicographicProduct, 44
LindgrenSousselierGraph, 67
LineDigraph, 47
LineUndirectedDigraph, 47
ListNamedDigraphs, 18
LollipopGraph, 67

MaximalAntiSymmetricSubdigraph, 25
MaximalAntiSymmetricSubdigraphAttr, 25
MaximalCommonSubdigraph, 174
MaximalSymmetricSubdigraph, 24
MaximalSymmetricSubdigraphAttr, 24
MaximalSymmetricSubdigraphWithoutLoops, 24
MaximalSymmetricSubdigraphWithoutLoopsAttr, 24
MinimalCommonSuperdigraph, 175
MobiusLadderGraph, 68
ModularProduct, 45
MonomorphismsDigraphs, 179
MonomorphismsDigraphsRepresentatives, 179
MycielskiGraph, 68

NautyAutomorphismGroup, 161
NautyCanonicalDigraph, 165
NautyCanonicalLabelling
   for a digraph, 163
   for a digraph and a list, 164
NonLowerSemimodularPair, 101
NonUpperSemimodularPair, 101
NrSpanningTrees, 122
NullDigraph, 60

OddGraph, 69
OntDigraphs
   for a digraph and a perm, 157
   for a digraph and a transformation, 157
OntMultiDigraphs, 158
   for a digraph, perm, and perm, 158
OntSetsDigraphs
   for a set of digraphs and a perm, 158
OntTuplesDigraphs
   for a list of digraphs and a perm, 158
OutDegreeOfVertex, 97
OutDegrees, 96
OutDegreeSequence, 96
OutDegreeSet, 96

OuterPlanarEmbedding, 128
OutNeighbors, 95
OutNeighborsMutableCopy, 95
OutNeighborsOfVertex, 98
OutNeighbours, 95
OutNeighboursMutableCopy, 95
OutNeighboursOfVertex, 98

PartialOrderDigraphJoinOfVertices, 101
PartialOrderDigraphMeetOfVertices, 101
PathGraph, 69
PermutationStarGraph, 70
PetersenGraph, 70
PlainTimeString, 218
PlanarEmbedding, 127
PrintString, 210
PrismGraph, 71

QueenGraph, 72
QueensGraph, 72
QuotientDigraph, 27

RandomDigraph, 50
RandomLattice, 52
RandomMultiDigraph, 51
RandomTournament, 52
ReadDigraphs, 212
ReadDIMACSDigraph, 219
ReadPlainTextDigraph, 218
ReducedDigraph, 24
ReducedDigraphAttr, 24
RepresentativeOutNeighbours, 172
RookGraph, 73
RooksGraph, 73

SemigroupOfCayleyDigraph, 123
SetDigraphEdgeLabel, 87
SetDigraphEdgeLabels
   for a digraph and a function, 88
   for a digraph and a list of lists, 88
SetDigraphVertexLabel, 85
SetDigraphVertexLabels, 86
Sparse6String, 211

Splash, 200
SquareGridGraph, 74
StackedBookGraph, 56
StackedPrismGraph, 72
StarGraph, 75
String, 210
StrongOrientation, 111
StrongOrientationAttr, 111
StrongProduct, 45
SubdigraphHomeomorphicToK23, 129
SubdigraphHomeomorphicToK33, 129
SubdigraphHomeomorphicToK4, 129
TadpoleGraph, 76
TCodeDecoder, 217
TournamentLineDecoder, 216
TriangularGridGraph, 75
UndirectedSpanningForest, 26
UndirectedSpanningForestAttr, 26
UndirectedSpanningTree, 26
UndirectedSpanningTreeAttr, 26
VerticesReachableFrom, 114
WalshHadamardGraph, 76
WebGraph, 77
WheelGraph, 77
WindmillGraph, 77
WriteDigraphs, 214
WriteDIMACSDigraph, 219
WritePlainTextDigraph, 218