# Examples-ForHomalg

## Examples for the GAP Package homalg

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# Introduction

[BLH20]

# Chapter 2

# Installation of the ExamplesForHomalg Package

To install this package just extract the package's archive file to the GAP pkg directory.

By default the ExamplesForHomalg package is not automatically loaded by GAP when it is installed. You must load the package with

```
LoadPackage("ExamplesForHomalg");
```

before its functions become available.

Please, send us an e-mail if you have any questions, remarks, suggestions, etc. concerning this package. Also, I would be pleased to hear about applications of this package.

Mohamed Barakat and Simon Görtzen.

# Chapter 3

# Examples

## 3.1 Spectral Filtrations

### 3.1.1 ExtExt

This is Example B.2 in [Bar09].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> wmat := HomalgMatrix( "[ \
> x*y,  y*z,    z,        0,         0,    \
> x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
> x^4,  x^3*z,  0,        x^2*z,     -x*z, \
> 0,    0,      x*y,      -y^2,      x^2-1,\
> 0,    0,      x^2*z,    -x*y*z,    y*z,  \
> 0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
> ]", 6, 5, Qxyz );
<A 6 x 5 matrix over an external ring>
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> Y := Hom( Qxyz, W );
<A right module on 5 generators satisfying yet unknown relations>
gap> SetInfoLevel( InfoWarning, 0 );
gap> F := InsertObjectInMultiFunctor( Functor_Hom_for_fp_modules, 2, Y, "TensorY" );
<The functor TensorY for f.p. modules and their maps over computable rings>
gap> SetInfoLevel( InfoWarning, 1 );
gap> G := LeftDualizingFunctor( Qxyz );;
gap> II_E := GrothendieckSpectralSequence( F, G, W );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:
```

```
 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s s s s
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 * * * *
 . * * *
 . . . *
---------
Level 2:

 * * s s
 * * * *
 . * * *
 . . . *
---------
Level 3:

 * s s s
 * s s s
 . . s *
 . . . *
---------
```

```
Level 4:

 s s s s
 . s s s
 . . s s
 . . . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<An ascending filtration with degrees [ -3 .. 0 ] and graded parts:

0:   <A non-zero left module presented by yet unknown relations for 23 generator\
s>
  -1:   <A non-zero left module presented by 37 relations for 22 generators>
  -2:   <A non-zero left module presented by 32 relations for 10 generators>
  -3:   <A non-zero left module presented by 33 relations for 5 generators>
of
<A non-zero left module presented by 117 relations for 37 generators>>
gap> ByASmallerPresentation( filt );
<An ascending filtration with degrees [ -3 .. 0 ] and graded parts:
   0:   <A non-zero left module presented by 26 relations for 16 generators>
  -1:   <A non-zero left module presented by 30 relations for 14 generators>
  -2:   <A non-zero left module presented by 18 relations for 7 generators>
  -3:   <A non-zero left module presented by 12 relations for 4 generators>
of
<A non-zero left module presented by 48 relations for 20 generators>>
gap> m := IsomorphismOfFiltration( filt );
<A non-zero isomorphism of left modules>
```

### 3.1.2 Purity

This is Example B.3 in [Bar09].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> wmat := HomalgMatrix( "[ \
> x*y,  y*z,    z,        0,         0,    \
> x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
> x^4,  x^3*z,  0,        x^2*z,     -x*z, \
> 0,    0,      x*y,      -y^2,      x^2-1,\
> 0,    0,      x^2*z,    -x*y*z,    y*z,  \
> 0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
> ]", 6, 5, Qxyz );
<A 6 x 5 matrix over an external ring>
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> filt := PurityFiltration( W );
<The ascending purity filtration with degrees [ -3 .. 0 ] and graded parts:

0:   <A codegree-[ 1, 1 ]-pure rank 2 left module presented by 3 relations for 4\
 generators>

-1:   <A codegree-1-pure grade 1 left module presented by 4 relations for 3 gene\
rators>
```

```
-2:   <A cyclic reflexively pure grade 2 left module presented by 2 relations fo\
r a cyclic generator>

-3:   <A cyclic reflexively pure grade 3 left module presented by 3 relations fo\
r a cyclic generator>
of
<A non-pure rank 2 left module presented by 6 relations for 5 generators>>
gap> W;
<A non-pure rank 2 left module presented by 6 relations for 5 generators>
gap> II_E := SpectralSequence( filt );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s . . .
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
```

```
---------
Level 1:

 * * * *
 * * * *
 . * * *
 . . . *
---------
Level 2:

 s . . .
 * s . .
 . * * .
 . . . *
---------
Level 3:

 s . . .
 * s . .
 . . s .
 . . . *
---------
Level 4:

 s . . .
 . s . .
 . . s .
 . . . s

gap> m := IsomorphismOfFiltration( filt );
<A non-zero isomorphism of left modules>
gap> IsIdenticalObj( Range( m ), W );
true
gap> Source( m );
<A left module presented by 12 relations for 9 generators (locked)>
gap> Display( last );
0,  0,x, -y,0,1, 0,    0,  0,
x*y,0,-z,0, 0,0, 0,    0,  0,
x^2,0,0, -z,1,0, 0,    0,  0,
0,  0,0, 0, y,-z,0,    0,  0,
0,  0,0, 0, 0,x, -y,   -1, 0,
0,  0,0, 0, x,0, -z,   0,  -1,
0,  0,0, 0, 0,-y,x^2-1,0,  0,
0,  0,0, 0, 0,0, 0,    z,  0,
0,  0,0, 0, 0,0, 0,    y-1,0,
0,  0,0, 0, 0,0, 0,    0,  z,
0,  0,0, 0, 0,0, 0,    0,  y,
0,  0,0, 0, 0,0, 0,    0,  x

Cokernel of the map

Q[x,y,z]^(1x12) --> Q[x,y,z]^(1x9),
```

```
currently represented by the above matrix
gap> Display( filt );
Degree 0:

0,  0,x, -y,
x*y,0,-z,0,
x^2,0,0, -z

Cokernel of the map

Q[x,y,z]^(1x3) --> Q[x,y,z]^(1x4),

currently represented by the above matrix
----------
Degree -1:

y,-z,0,
0,x, -y,
x,0, -z,
0,-y,x^2-1

Cokernel of the map

Q[x,y,z]^(1x4) --> Q[x,y,z]^(1x3),

currently represented by the above matrix
----------
Degree -2:

Q[x,y,z]/< z, y-1 >
----------
Degree -3:

Q[x,y,z]/< z, y, x >
gap> Display( m );
1,   0,     0,  0,   0,
0,   1,     0,  0,   0,
0,   -y,    -1, 0,   0,
0,   -x,    0,  -1,  0,
-x^2,-x*z,  0,  -z,  0,
0,   0,     x,  -y,  0,
0,   0,     0,  0,   -1,
0,   0,     x^2,-x*y,y,
-x^3,-x^2*z,0,  -x*z,z

the map is currently represented by the above 9 x 5 matrix
```

### 3.1.3 A3_Purity

This is Example B.4 in [Bar09].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> A3 := RingOfDerivations( Qxyz, "Dx,Dy,Dz" );
Q[x,y,z]<Dx,Dy,Dz>
gap> nmat := HomalgMatrix( "[ \
> 3*Dy*Dz-Dz^2+Dx+3*Dy-Dz,           3*Dy*Dz-Dz^2,     \
> Dx*Dz+Dz^2+Dz,                     Dx*Dz+Dz^2,       \
> Dx*Dy,                             0,                \
> Dz^2-Dx+Dz,                        3*Dx*Dy+Dz^2,     \
> Dx^2,                              0,                \
> -Dz^2+Dx-Dz,                       3*Dx^2-Dz^2,      \
> Dz^3-Dx*Dz+Dz^2,                   Dz^3,             \
> 2*x*Dz^2-2*x*Dx+2*x*Dz+3*Dx+3*Dz+3,2*x*Dz^2+3*Dx+3*Dz\
> ]", 8, 2, A3 );
<A 8 x 2 matrix over an external ring>
gap> N := LeftPresentation( nmat );
<A left module presented by 8 relations for 2 generators>
gap> filt := PurityFiltration( N );
<The ascending purity filtration with degrees [ -3 .. 0 ] and graded parts:
   0:   <A zero left module>

-1:   <A cyclic reflexively pure grade 1 left module presented by 1 relation for\
 a cyclic generator>

-2:   <A cyclic reflexively pure grade 2 left module presented by 2 relations fo\
r a cyclic generator>

-3:   <A cyclic reflexively pure grade 3 left module presented by 3 relations fo\
r a cyclic generator>
of
<A non-pure grade 1 left module presented by 8 relations for 2 generators>>
gap> II_E := SpectralSequence( filt );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 2 ] each consisting of left modules at bidegrees [ -4 .. 0 ]x
[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -4 .. 0 ] ]
---------
Level 0:

 * * * *
 . * * *
 . . * *
 . . . *
 . . . *
---------
Level 1:

 * * * *
```

```
 . . . .
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s . . .
 . . . .
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -4 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * * *
 . . * * *
 . . . * *
 . . . . *
---------
Level 1:

 * * * * *
 . . * * *
 . . . * *
 . . . . .
---------
Level 2:

 . s . . .
 . . s . .
 . . . s .
 . . . . .
gap> m := IsomorphismOfFiltration( filt );
<A non-zero isomorphism of left modules>
gap> IsIdenticalObj( Range( m ), N );
true
gap> Source( m );
<A left module presented by 6 relations for 3 generators (locked)>
gap> Display( last );
Dx,1/3,1/216*x,
0, Dy, -1/144,
0, Dx, 1/48,
0, 0,  Dz,
0, 0,  Dy,
0, 0,  Dx
```

```
Cokernel of the map

R^(1x6) --> R^(1x3), ( for R := Q[x,y,z]<Dx,Dy,Dz> )

currently represented by the above matrix
gap> Display( filt );
Degree 0:

0
----------
Degree -1:

Q[x,y,z]<Dx,Dy,Dz>/< Dx >
----------
Degree -2:

Q[x,y,z]<Dx,Dy,Dz>/< Dy, Dx >
----------
Degree -3:

Q[x,y,z]<Dx,Dy,Dz>/< Dz, Dy, Dx >
gap> Display( m );
1,                      1,
3*Dz+3,                 3*Dz,
144*Dz^2-144*Dx+144*Dz,144*Dz^2

the map is currently represented by the above 3 x 2 matrix
```

### 3.1.4 TorExt-Grothendieck

This is Example B.5 in [Bar09].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> wmat := HomalgMatrix( "[ \
> x*y,  y*z,    z,        0,         0,    \
> x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
> x^4,  x^3*z,  0,        x^2*z,     -x*z, \
> 0,    0,      x*y,      -y^2,      x^2-1,\
> 0,    0,      x^2*z,    -x*y*z,    y*z,  \
> 0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
> ]", 6, 5, Qxyz );
<A 6 x 5 matrix over an external ring>
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> F := InsertObjectInMultiFunctor( Functor_TensorProduct_for_fp_modules, 2, W, "TensorW" );
<The functor TensorW for f.p. modules and their maps over computable rings>
gap> G := LeftDualizingFunctor( Qxyz );;
gap> II_E := GrothendieckSpectralSequence( F, G, W );
<A stable cohomological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ]>
```

```
gap> Display( II_E );
The associated transposed spectral sequence:

a cohomological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s s s s
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a cohomological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 * * * *
 . * * *
 . . . *
---------
Level 2:

 * * s s
 * * * *
 . * * *
 . . . *
```

```
---------
Level 3:

 * s s s
 . s s s
 . . s *
 . . . s
---------
Level 4:

 s s s s
 . s s s
 . . s s
 . . . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:

-3:   <A non-zero cyclic torsion left module presented by yet unknown relations \
for a cyclic generator>
  -2:   <A non-zero left module presented by 17 relations for 6 generators>
  -1:   <A non-zero left module presented by 28 relations for 12 generators>
   0:   <A non-zero left module presented by 13 relations for 10 generators>
of
<A left module presented by yet unknown relations for 49 generators>>
gap> ByASmallerPresentation( filt );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:

-3:   <A non-zero cyclic torsion left module presented by 3 relations for a cycl\
ic generator>
  -2:   <A non-zero left module presented by 12 relations for 4 generators>
  -1:   <A non-zero left module presented by 21 relations for 8 generators>
   0:   <A non-zero left module presented by 11 relations for 10 generators>
of
<A non-zero left module presented by 27 relations for 14 generators>>
gap> m := IsomorphismOfFiltration( filt );
<A non-zero isomorphism of left modules>
```

### 3.1.5 TorExt

This is Example B.6 in [Bar09].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> wmat := HomalgMatrix( "[ \
> x*y,  y*z,    z,        0,         0,    \
> x^3*z,x^2*z^2,0,        x*z^2,     -z^2, \
> x^4,  x^3*z,  0,        x^2*z,     -x*z, \
> 0,    0,      x*y,      -y^2,      x^2-1,\
> 0,    0,      x^2*z,    -x*y*z,    y*z,  \
> 0,    0,      x^2*y-x^2,-x*y^2+x*y,y^2-y \
> ]", 6, 5, Qxyz );
<A 6 x 5 matrix over an external ring>
```

```
gap> W := LeftPresentation( wmat );
<A left module presented by 6 relations for 5 generators>
gap> P := Resolution( W );
<A right acyclic complex containing 3 morphisms of left modules at degrees
[ 0 .. 3 ]>
gap> GP := Hom( P );
<A cocomplex containing 3 morphisms of right modules at degrees [ 0 .. 3 ]>
gap> FGP := GP * P;
<A cocomplex containing 3 morphisms of left complexes at degrees [ 0 .. 3 ]>
gap> BC := HomalgBicomplex( FGP );
<A bicocomplex containing left modules at bidegrees [ 0 .. 3 ]x[ -3 .. 0 ]>
gap> p_degrees := ObjectDegreesOfBicomplex( BC )[1];
[ 0 .. 3 ]
gap> II_E := SecondSpectralSequenceWithFiltration( BC, p_degrees );
<A stable cohomological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 3 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a cohomological spectral sequence at bidegrees
[ [ 0 .. 3 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * * *
 * * * *
 * * * *
 * * * *
---------
Level 1:

 * * * *
 . . . .
 . . . .
 . . . .
---------
Level 2:

 s s s s
 . . . .
 . . . .
 . . . .

Now the spectral sequence of the bicomplex:

a cohomological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 3 ] ]
---------
Level 0:

 * * * *
```

```
 * * * *
 * * * *
 * * * *
---------
Level 1:

 * * * *
 * * * *
 * * * *
 * * * *
---------
Level 2:

 * * s s
 * * * *
 . * * *
 . . . *
---------
Level 3:

 * s s s
 . s s s
 . . s *
 . . . s
---------
Level 4:

 s s s s
 . s s s
 . . s s
 . . . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:

-3:   <A non-zero cyclic torsion left module presented by yet unknown relations \
for a cyclic generator>
  -2:   <A non-zero left module presented by 15 relations for 6 generators>
  -1:   <A non-zero left module presented by 29 relations for 13 generators>
   0:   <A non-zero left module presented by 13 relations for 10 generators>
of
<A left module presented by yet unknown relations for 31 generators>>
gap> ByASmallerPresentation( filt );
<A descending filtration with degrees [ -3 .. 0 ] and graded parts:

-3:   <A non-zero cyclic torsion left module presented by 3 relations for a cycl\
ic generator>
  -2:   <A non-zero left module presented by 11 relations for 4 generators>
  -1:   <A non-zero left module presented by 22 relations for 8 generators>
   0:   <A non-zero left module presented by 11 relations for 10 generators>
of
<A non-zero left module presented by 24 relations for 12 generators>>
gap> m := IsomorphismOfFiltration( filt );
```

```
<A non-zero isomorphism of left modules>
```

### 3.1.6 CodegreeOfPurity

This is Example B.7 in [Bar09].

Example

```
gap> Qxyz := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z";
Q[x,y,z]
gap> vmat := HomalgMatrix( "[ \
> 0,  0,  x,-z, \
> x*z,z^2,y,0,  \
> x^2,x*z,0,y   \
> ]", 3, 4, Qxyz );
<A 3 x 4 matrix over an external ring>
gap> V := LeftPresentation( vmat );
<A non-torsion left module presented by 3 relations for 4 generators>
gap> wmat := HomalgMatrix( "[ \
> 0,  0,  x,-y, \
> x*y,y*z,z,0,  \
> x^2,x*z,0,z   \
> ]", 3, 4, Qxyz );
<A 3 x 4 matrix over an external ring>
gap> W := LeftPresentation( wmat );
<A non-torsion left module presented by 3 relations for 4 generators>
gap> Rank( V );
2
gap> Rank( W );
2
gap> ProjectiveDimension( V );
2
gap> ProjectiveDimension( W );
2
gap> DegreeOfTorsionFreeness( V );
1
gap> DegreeOfTorsionFreeness( W );
1
gap> CodegreeOfPurity( V );
[ 2 ]
gap> CodegreeOfPurity( W );
[ 1, 1 ]
gap> filtV := PurityFiltration( V );
<The ascending purity filtration with degrees [ -2 .. 0 ] and graded parts:

0:   <A codegree-[ 2 ]-pure rank 2 left module presented by 3 relations for 4 ge\
nerators>
  -1:   <A zero left module>
  -2:   <A zero left module>
of
<A codegree-[ 2 ]-pure rank 2 left module presented by 3 relations for 4 gener\
ators>>
gap> filtW := PurityFiltration( W );
<The ascending purity filtration with degrees [ -2 .. 0 ] and graded parts:
```

```
0:   <A codegree-[ 1, 1 ]-pure rank 2 left module presented by 3 relations for 4\
 generators>
  -1:   <A zero left module>
  -2:   <A zero left module>
of
<A codegree-[ 1, 1 ]-pure rank 2 left module presented by 3 relations for 4 ge\
nerators>>
gap> II_EV := SpectralSequence( filtV );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 2 ]>
gap> Display( II_EV );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 2 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * *
 * * *
 * * *
 . * *
---------
Level 1:

 * * *
 . . .
 . . .
 . . .
---------
Level 2:

 s . .
 . . .
 . . .
 . . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 2 ] ]
---------
Level 0:

 * * * *
 * * * *
 . * * *
---------
Level 1:
```

```
 * * * *
 * * * *
 . . * *
---------
Level 2:

 * . . .
 * . . .
 . . * *
---------
Level 3:

 * . . .
 . . . .
 . . . *
---------
Level 4:

 . . . .
 . . . .
 . . . s
gap> II_EW := SpectralSequence( filtW );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 4 ] each consisting of left modules at bidegrees [ -3 .. 0 ]x
[ 0 .. 2 ]>
gap> Display( II_EW );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 2 ], [ -3 .. 0 ] ]
---------
Level 0:

 * * *
 * * *
 . * *
 . . *
---------
Level 1:

 * * *
 . . .
 . . .
 . . .
---------
Level 2:

 s . .
 . . .
 . . .
 . . .
```

```
Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -3 .. 0 ], [ 0 .. 2 ] ]
---------
Level 0:

 * * * *
 . * * *
 . . * *
---------
Level 1:

 * * * *
 . * * *
 . . . *
---------
Level 2:

 * . . .
 . * . .
 . . . *
---------
Level 3:

 * . . .
 . . . .
 . . . *
---------
Level 4:

 . . . .
 . . . .
 . . . s
```

### 3.1.7 HomHom

This corresponds to the example of Section 2 in [BR06].

Example

```
gap> R := HomalgRingOfIntegersInExternalGAP( ) / 2^8;
Z/( 256 )
gap> Display( R );
<A residue class ring>
gap> M := LeftPresentation( [ 2^5 ], R );
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> Display( M );
Z/( 256 )/< |[ 32 ]| >
gap> M;
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> _M := LeftPresentation( [ 2^3 ], R );
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> Display( _M );
```

```
Z/( 256 )/< |[ 8 ]| >
gap> _M;
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> alpha2 := HomalgMap( [ 1 ], M, _M );
<A "homomorphism" of left modules>
gap> IsMorphism( alpha2 );
true
gap> alpha2;
<A homomorphism of left modules>
gap> Display( alpha2 );
[ [  1 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
gap> M_ := Kernel( alpha2 );
<A cyclic left module presented by yet unknown relations for a cyclic generato\
r>
gap> alpha1 := KernelEmb( alpha2 );
<A monomorphism of left modules>
gap> seq := HomalgComplex( alpha2 );
<An acyclic complex containing a single morphism of left modules at degrees
[ 0 .. 1 ]>
gap> Add( seq, alpha1 );
gap> seq;
<A sequence containing 2 morphisms of left modules at degrees [ 0 .. 2 ]>
gap> IsShortExactSequence( seq );
true
gap> seq;
<A short exact sequence containing 2 morphisms of left modules at degrees
[ 0 .. 2 ]>
gap> Display( seq );
-------------------------
at homology degree: 2
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  24 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 1
Z/( 256 )/< |[ 32 ]| >
-------------------------
[ [  1 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 0
```

```
Z/( 256 )/< |[ 8 ]| >
-------------------------
gap> K := LeftPresentation( [ 2^7 ], R );
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> L := RightPresentation( [ 2^4 ], R );
<A cyclic right module on a cyclic generator satisfying 1 relation>
gap> triangle := LHomHom( 4, seq, K, L, "t" );
<An exact triangle containing 3 morphisms of left complexes at degrees
[ 1, 2, 3, 1 ]>
gap> lehs := LongSequence( triangle );
<A sequence containing 14 morphisms of left modules at degrees [ 0 .. 14 ]>
gap> ByASmallerPresentation( lehs );
<A non-zero sequence containing 14 morphisms of left modules at degrees
[ 0 .. 14 ]>
gap> IsExactSequence( lehs );
false
gap> lehs;
<A non-zero left acyclic complex containing
14 morphisms of left modules at degrees [ 0 .. 14 ]>
gap> Assert( 0, IsLeftAcyclic( lehs ) );
gap> Display( lehs );
-------------------------
at homology degree: 14
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  4 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 13
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 12
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 11
Z/( 256 )/< |[ 4 ]| >
-------------------------
```

```
[ [  4 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 10
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 9
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 8
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  4 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 7
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 6
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 5
```

```
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  4 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 4
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 3
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 2
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  8 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 1
Z/( 256 )/< |[ 16 ]| >
-------------------------
[ [  1 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 0
Z/( 256 )/< |[ 8 ]| >
-------------------------
```

## 3.2 Commutative Algebra

### 3.2.1 Eliminate

Example

```
gap> R := HomalgFieldOfRationalsInDefaultCAS( ) * "x,y,z,l,m";
Q[x,y,z,l,m]
gap> var := Indeterminates( R );
[ x, y, z, l, m ]
gap> x := var[1];; y := var[2];; z := var[3];; l := var[4];; m := var[5];;
gap> L := [ x*m+l-4, y*m+l-2, z*m-l+1, x^2+y^2+z^2-1, x+y-z ];
[ x*m+l-4, y*m+l-2, z*m-l+1, x^2+y^2+z^2-1, x+y-z ]
gap> e := Eliminate( L, [ l, m ] );
<A non-zero right regular 3 x 1 matrix over an external ring>
gap> Display( e );
4*y+z,
4*x-5*z,
21*z^2-8
gap> I := LeftSubmodule( e );
<A torsion-free (left) ideal given by 3 generators>
gap> Display( I );
4*y+z,
4*x-5*z,
21*z^2-8

A (left) ideal generated by the 3 entries of the above matrix
gap> J := LeftSubmodule( "x+y-z, -2*z-3*y+x, x^2+y^2+z^2-1", R );
<A torsion-free (left) ideal given by 3 generators>
gap> I = J;
true
```

# References


[Bar09] Mohamed Barakat. Spectral filtrations via generalized morphisms. (arXiv:0904.0240), 2009. 6, 8, 11, 14, 16, 19

[BLH20] Mohamed Barakat and Markus Lange-Hegermann. *The* homalg *package – A homological algebra* GAP4 *meta-package for computable Abelian categories*, 2007–2020. (https://homalg-project.github.io/pkg/homalg). 4

[BR06] Mohamed Barakat and Daniel Robertz. homalg: First steps to an abstract package for homological algebra. In *Proceedings of the X meeting on computational algebra and its applications - EACA 2006*, pages 29–32, Sevilla, Spain, September 2006. 22

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