Modules

A homalg based package for the Abelian category of finitely presented modules over computable rings

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Thomas Bächler

Mohamed Barakat

Florian Diebold

Sebastian Gutsche

Markus Lange-Hegermann

Vinay Wagh

Thomas Bächler
Email: thomas@momo.math.rwth-aachen.de
Homepage: http://wwwb.math.rwth-aachen.de/~thomas/
Address: Thomas Bächler
Lehrstuhl B für Mathematik
RWTH Aachen
Templergraben 64
52062 Aachen
Germany

Mohamed Barakat
Email: mohamed.barakat@uni-siegen.de
Homepage: https://mohamed-barakat.github.io
Address: Walter-Flex-Str. 3
57072 Siegen
Germany

Florian Diebold
Email: diebold@mathematik.uni-kl.de
Address: Department of Mathematics
University of Kaiserslautern
67653 Kaiserslautern
Germany

Sebastian Gutsche
Email: gutsche@mathematik.uni-siegen.de
Homepage: https://sebasguts.github.io
Address: Department Mathematik
Universität Siegen
Walter-Flex-Straße 3
57072 Siegen
Germany

Markus Lange-Hegermann
Email: markus.lange-hegermann@hs-owl.de
Homepage: https://www.th-owl.de/eecs/fachbereich/team/markus-lange-hegermann/
Address: Markus Lange-Hegermann
Hochschule Ostwestfalen-Lippe
Liebigstraße 87
32657 Lemgo
Germany

Vinay Wagh
Email: waghoba@gmail.com
Homepage: http://www.iitg.ernet.in/vinay.wagh/

Address: E-102, Department of Mathematics,
Indian Institute of Technology Guwahati,
Guwahati, Assam, India.
PIN: 781 039.
India

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Chapter 1

Introduction

1.1 What is the role of the Modules package in the homalg project?

1.1.1 Modules provides ...

It provides procedures to construct basic objects in homological algebra:

- modules (generators, relations)
- submodules (as images of maps)
- maps

Beside these so-called constructors Modules provides *operations* to perform computations with these objects. The list of operations includes:

- resolution of modules
- images of maps
- the functors `Hom` and `TensorProduct` (`Ext` and `Tor` are then provided by homalg)
- test if a module is torsion-free, reflexive, projective, stably free, free, pure
- determine the rank, grade, projective dimension, degree of torsion-freeness, and codegree of purity of a module

Using the philosophy of GAP4, one or more methods are *installed* for each operation, depending on *properties* and *attributes* of these objects. These properties and attributes can themselves be computed by methods installed for this purpose.

1.1.2 Rings supported in a sufficient way

Through out this manual the following terminology is used. We say that a computer algebra system "sufficiently supports" a ring R, if it contains procedures to effectively solve one-sided inhomogeneous linear systems $XA = B$ and $AX = B$ with coefficients over R ($\rightarrow$ 1.1.3).

1.1.3 Principal limitation

Note that the solution space of the one-sided finite dimensional system $YA = 0$ (resp. $AY = 0$) over a left (resp. right) noetherian ring R is a finitely generated left (resp. right) R-module, even if R is not commutative. The solution space of the linear system $X_1A_1 + A_2X_2 + A_3X_3A_4 = 0$ is in general not an R-module, and worse, in general not finitely generated over the center of R. Modules can only handle homological problems that lead to *one sided finite dimensional* homogeneous or inhomogeneous systems over the underlying ring R. Such problems are called problems of *finite type* over R. Typically, the computation of $\mathtt{Hom}(M,N)$ of two (even) finitely generated modules over a *non*commutative ring R is generally *not* of finite type over R, unless at least one of the two modules is an R-bimodule. Also note that over a commutative ring any linear system can be easily brought to a one-sided form. For more details see [BR08].

1.1.4 Ring dictionaries (technical)

Modules uses the so-called `homalgTable`, which is stored in the ring, to know how to delegate the necessary matrix operations. I.e. the `homalgTable` serves as a small dictionary that enables Modules to speak (as much as needed of) the language of the computer algebra system which hosts the ring and the matrices. The GAP internal ring of integers is the only ring which Modules endows with a `homalgTable`. Other packages like GaussForHomalg and RingsForHomalg provide dictionaries for further rings. While GaussForHomalg defines internal rings and matrices, the package RingsForHomalg enables defining external rings and matrices in a wide range of (external) computer algebra systems (Singular, Sage, Macaulay2, MAGMA, Maple) by providing appropriate dictionaries.

Since these dictionaries are all what is needed to handle matrix operations, Modules does not distinguish between handling internal and handling external matrices. Even the physical communication with the external systems is not at all a concern of Modules. This is the job of the package IO_ForHomalg, which is based on the powerful IO package of Max Neunhöffer. Furthermore, for all structures beyond matrices (from relations, generators, and modules, to functors and spectral sequences) Modules no longer distinguishes between internal and external.

1.1.5 The advantages of the outsourcing concept

Linking different systems to achieve one task is a highly attractive idea, especially if it helps to avoid reinventing wheels over and over again. This was essential for homalg, since Singular and MAGMA provide the fastest and most advanced Gröbner basis algorithms, while GAP4 is by far the most convenient programming language to realize complex mathematical structures (→ Appendix (**homalg: Why GAP4?**)). Second, the implementation of the homological constructions is automatically universal, since it is independent of where the matrices reside and how the several matrix operations are realized. In particular, homalg will always be able to use the system with the fastest Gröbner basis implementation. In this respect is homalg and all packages that build upon it future proof.

1.1.6 Does this mean that homalg has only algorithms for the generic case?

No, on the contrary. There are a lot of specialized algorithms installed in homalg. These algorithms are based on properties and attributes that – thanks to GAP4 – homalg objects can carry (→ Appendix (**homalg: GAP4 is a mathematical object-oriented programming language**)): Not only can homalg take the special nature of the underlying ring into account, it also deals with modules, complexes, ... depending on their special properties. Still, these special algorithms, like all algorithms in homalg, are independent of the computer algebra system which hosts the matrices and which will perform the several matrix operations.

1.1.7 The principle of least communication (technical)

Linking different systems can also be highly problematic. The following two points are often among the major sources of difficulties:

- Different systems use different languages:
  It takes a huge amount of time and effort to teach systems the dialects of each others. These dialects are also rarely fixed forever, and might very well be subject to slight modifications. So the larger the dictionary, the more difficult is its maintenance.

- Data has to be transferred from one system to another:
  Even if there is a unified data format, transferring data between systems can lead to performance losses, especially when a big amount of data has to be transferred.

Solving these two difficulties is an important part of Modules's design. Modules splits homological computations into two parts. The matrices reside in a system which provides fast matrix operations (addition, multiplication, bases and normal form computations), while the higher structures (modules, maps, complexes, chain morphisms, spectral sequences, functors, ...) with their properties, attributes, and algorithms live in GAP4, as the system where one can easily create such complex structures and handle all their logical dependencies. With this split there is no need to transfer each sort of data outside of its system. The remaining communication between GAP4 and the system hosting the matrices gets along with a tiny dictionary. Moreover, GAP4, as it manages and delegates all computations, also manages the whole data flow, while the other system does not even recognize that it is part of a bidirectional communication.

The existence of such a clear cut is certainly to some extent due to the special nature of homological computations.

1.1.8 Frequently asked questions

- Q: Does outsourcing the matrices mean that Modules is able to compute spectral sequences, for example, without ever seeing the matrices involved in the computation?

  A: Yes.

- Q: Can Modules profit from the implementation of homological constructions like `Hom`, `Ext`, ... in Singular?

  A: No. This is for a lot of reasons incompatible with the idea and design ($\rightarrow$ 1) of Modules.

- Q: Are the external systems involved in the higher algorithms?

  A: No. They host all the matrices and do all matrix operations delegated to them without knowing what for. The meaning of the matrices and their logical interrelation is only known to GAP4.

- Q: Do developers of packages building upon Modules need to know anything about the communication with the external systems?

  A: No, unless they want to use more features of the external systems than those reflected by Modules. For this purpose, developers can use the unified communication interface provideb by HomalgToCAS. This is the interface used by Modules.

1.2 This manual

Chapter 2 describes the installation of this package, while Chapter 3 provides a short quick guide to build your first own example, using the package ExamplesForHomalg. The remaining chapters are each devoted to one of the homalg objects (→ 1.1.1) with its constructors, properties, attributes, and operations.

Chapter 2

Installation of the Modules Package

To install this package just extract the package's archive file to the GAP `pkg` directory.

By default the Modules package is not automatically loaded by GAP when it is installed. You must load the package with

`LoadPackage`("Modules");

before its functions become available.

Please, send me an e-mail if you have any questions, remarks, suggestions, etc. concerning this package. Also, I would be pleased to hear about applications of this package.

Mohamed Barakat and Markus Lange-Hegermann

Chapter 3

Quick Start

This chapter should give you a quick guide to create your first example in homalg.

3.1 Why are all examples in this manual over $\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$?

As the reader might notice, all examples in this manual will be either over $\mathbb{Z}$ or over one of its residue class rings $\mathbb{Z}/m\mathbb{Z}$. There are two reasons for this. The first reason is that GAP does not natively support rings other than $\mathbb{Z}$ in a *sufficient* way ($\rightarrow$ 1.1.2).

The second and more important reason is to underline the fact the all effective homological constructions that are relevant for Modules have only as much to do with the Gröbnerbasis algorithm as they do with the Hermite algorithm for the ring $\mathbb{Z}$; both algorithms are used to effectively solve inhomogeneous linear systems over the respective ring. And Modules is designed to use rings and matrices over these rings together with all their operations as a black box. In other words: Because Modules works for $\mathbb{Z}$, it works by its design for all other computable rings.

3.2 `gap> ExamplesForHomalg();`

To quickly create a ring for use with Modules enter

```
ExamplesForHomalg();
```

which will load the package ExamplesForHomalg (if installed) and provide a step by step guide to create the ring. For the full core functionality you need to install the packages homalg, HomalgToCAS, IO_ForHomalg, RingsForHomalg, Gauss, and GaussForHomalg. They are part of the homalg project.

3.3 A typical example

3.3.1 HomHom

The following example is taken from Section 2 of [BR06].

The computation takes place over the residue class ring $R = \mathbb{Z}/2^8\mathbb{Z}$ using the generic support for residue class rings provided by the subpackage ResidueClassRingForHomalg of the MatricesForHomalg package. For a native support of the rings $R = \mathbb{Z}/p^n\mathbb{Z}$ use the GaussForHomalg package.

Here we compute the (infinite) long exact homology sequence of the covariant functor $Hom(Hom(-,\mathbb{Z}/2^7\mathbb{Z}),\mathbb{Z}/2^4\mathbb{Z})$ (and its left derived functors) applied to the short exact sequence

$$0 \longrightarrow M\_ = \mathbb{Z}/2^2\mathbb{Z} \xrightarrow{\alpha_1} M = \mathbb{Z}/2^5\mathbb{Z} \xrightarrow{\alpha_2} \_M = \mathbb{Z}/2^3\mathbb{Z} \longrightarrow 0\,.$$

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> Display( ZZ );
<An internal ring>
gap> R := ZZ / 2^8;
Z/( 256 )
gap> Display( R );
<A residue class ring>
gap> M := LeftPresentation( [ 2^5 ], R );
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> Display( M );
Z/( 256 )/< |[ 32 ]| >
gap> _M := LeftPresentation( [ 2^3 ], R );
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> Display( _M );
Z/( 256 )/< |[ 8 ]| >
gap> alpha2 := HomalgMap( [ 1 ], M, _M );
<A "homomorphism" of left modules>
gap> IsMorphism( alpha2 );
true
gap> alpha2;
<A homomorphism of left modules>
gap> Display( alpha2 );
[ [  1 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
gap> M_ := Kernel( alpha2 );
<A cyclic left module presented by yet unknown relations for a cyclic generato\
r>
gap> alpha1 := KernelEmb( alpha2 );
<A monomorphism of left modules>
gap> seq := HomalgComplex( alpha2 );
<An acyclic complex containing a single morphism of left modules at degrees
[ 0 .. 1 ]>
gap> Add( seq, alpha1 );
gap> seq;
<A sequence containing 2 morphisms of left modules at degrees [ 0 .. 2 ]>
gap> IsShortExactSequence( seq );
true
gap> seq;
<A short exact sequence containing 2 morphisms of left modules at degrees
[ 0 .. 2 ]>
```

```
gap> Display( seq );
-------------------------
at homology degree: 2
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  24 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 1
Z/( 256 )/< |[ 32 ]| >
-------------------------
[ [  1 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 0
Z/( 256 )/< |[ 8 ]| >
-------------------------
gap> K := LeftPresentation( [ 2^7 ], R );
<A cyclic left module presented by 1 relation for a cyclic generator>
gap> L := RightPresentation( [ 2^4 ], R );
<A cyclic right module on a cyclic generator satisfying 1 relation>
gap> triangle := LHomHom( 4, seq, K, L, "t" );
<An exact triangle containing 3 morphisms of left complexes at degrees
[ 1, 2, 3, 1 ]>
gap> lehs := LongSequence( triangle );
<A sequence containing 14 morphisms of left modules at degrees [ 0 .. 14 ]>
gap> ByASmallerPresentation( lehs );
<A non-zero sequence containing 14 morphisms of left modules at degrees
[ 0 .. 14 ]>
gap> IsExactSequence( lehs );
false
gap> lehs;
<A non-zero left acyclic complex containing
14 morphisms of left modules at degrees [ 0 .. 14 ]>
gap> Assert( 0, IsLeftAcyclic( lehs ) );
gap> Display( lehs );
-------------------------
at homology degree: 14
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  4 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
```

```
at homology degree: 13
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 12
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 11
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  4 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 10
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 9
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 8
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  4 ] ]

modulo [ 256 ]
```

```
the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 7
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 6
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 5
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  4 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 4
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 3
Z/( 256 )/< |[ 8 ]| >
-------------------------
[ [  2 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 2
Z/( 256 )/< |[ 4 ]| >
-------------------------
[ [  8 ] ]
```

```
modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 1
Z/( 256 )/< |[ 16 ]| >
-------------------------
[ [  1 ] ]

modulo [ 256 ]

the map is currently represented by the above 1 x 1 matrix
------------v------------
at homology degree: 0
Z/( 256 )/< |[ 8 ]| >
-------------------------
```

Chapter 4

Ring Maps

4.1 Ring Maps: Attributes

4.1.1 KernelSubobject (for ring maps)

▷ `KernelSubobject(phi)` (attribute)

Returns: a homalg submodule

The kernel ideal of the ring map *phi*.

4.1.2 KernelEmb (for ring maps)

▷ `KernelEmb(phi)` (attribute)

Returns: a homalg map

The embedding of the kernel ideal `Kernel`(*phi*) into the `Source`(*phi*), both viewed as modules over the ring R :=`Source`(*phi*) (cf. `Kernel` (4.2.1)).

4.2 Ring Maps: Operations and Functions

4.2.1 Kernel (for ring maps)

▷ `Kernel(phi)` (method)

Returns: a homalg module

The kernel ideal of the ring map *phi* as an abstract module.

Chapter 5

Relations

A finite presentation of a module is given by a finite set of generators and a finite set of relations among these generators. In homalg a set of relations of a left/right module is given by a matrix *rel*, the rows/columns of which are interpreted as relations among n generators, n being the number of columns/rows of the matrix *rel*.

The data structure of a module in homalg is designed to contain not only one but several sets of relations (together with corresponding sets of generators ($\rightarrow$ Chapter 6)). The different sets of relations are linked with so-called transition matrices ($\rightarrow$ Chapter 7).

The relations of a homalg module are evaluated in a lazy way. This avoids unnecessary computations.

5.1 Relations: Categories and Representations

5.1.1 IsHomalgRelations

▷ `IsHomalgRelations(rel)` (Category)

Returns: `true` or `false`

The GAP category of homalg relations.

5.1.2 IsHomalgRelationsOfLeftModule

▷ `IsHomalgRelationsOfLeftModule(rel)` (Category)

Returns: `true` or `false`

The GAP category of homalg relations of a left module.

(It is a subcategory of the GAP category `IsHomalgRelations`.)

5.1.3 IsHomalgRelationsOfRightModule

▷ `IsHomalgRelationsOfRightModule(rel)` (Category)

Returns: `true` or `false`

The GAP category of homalg relations of a right module.

(It is a subcategory of the GAP category `IsHomalgRelations`.)

5.1.4 IsRelationsOfFinitelyPresentedModuleRep

▷ `IsRelationsOfFinitelyPresentedModuleRep(rel)` (Representation)

Returns: `true` or `false`

The GAP representation of a finite set of relations of a finitely presented homalg module. (It is a representation of the GAP category `IsHomalgRelations` (5.1.1))

5.2 Relations: Constructors

5.3 Relations: Properties

5.3.1 CanBeUsedToDecideZeroEffectively

▷ `CanBeUsedToDecideZeroEffectively(rel)` (property)

Returns: `true` or `false`

Check if the homalg set of relations *rel* can be used for normal form reductions.

(no method installed)

5.3.2 IsInjectivePresentation

▷ `IsInjectivePresentation(rel)` (property)

Returns: `true` or `false`

Check if the homalg set of relations *rel* has zero syzygies.

5.4 Relations: Attributes

5.5 Relations: Operations and Functions

Chapter 6

Generators

To present a left/right module it suffices to take a matrix *rel* and interpret its rows/columns as relations among *n abstract* generators, where *n* is the number of columns/rows of *rel*. Only that these abstract generators are useless when it comes to specific modules like modules of homomorphisms, where one expects the generators to be maps between modules. For this reason a presentation of a module in homalg is not merely a matrix of relations, but together with a set of generators.

In homalg a set of generators of a left/right module is given by a matrix *gen* with rows/columns being interpreted as the generators.

The data structure of a module in homalg is designed to contain not only one but several sets of generators (together with their sets of relations (→ Chapter 5)). The different sets of generators are linked with so-called transition matrices (→ Chapter 7).

6.1 Generators: Categories and Representations

6.1.1 IsHomalgGenerators

▷ `IsHomalgGenerators(rel)` (Category)

Returns: `true` or `false`

The GAP category of homalg generators.

6.1.2 IsHomalgGeneratorsOfLeftModule

▷ `IsHomalgGeneratorsOfLeftModule(rel)` (Category)

Returns: `true` or `false`

The GAP category of homalg generators of a left module.

(It is a subcategory of the GAP category `IsHomalgGenerators`.)

6.1.3 IsHomalgGeneratorsOfRightModule

▷ `IsHomalgGeneratorsOfRightModule(rel)` (Category)

Returns: `true` or `false`

The GAP category of homalg generators of a right module.

(It is a subcategory of the GAP category `IsHomalgGenerators`.)

6.1.4 IsGeneratorsOfModuleRep

▷ `IsGeneratorsOfModuleRep(rel)` (Representation)

Returns: `true` or `false`

The GAP representation of a finite set of generators of a homalg module.
(It is a representation of the GAP category `IsHomalgGenerators` (6.1.1))

Code
```
DeclareRepresentation( "IsGeneratorsOfModuleRep",
        IsHomalgGenerators,
        [ "generators" ] );
```

6.1.5 IsGeneratorsOfFinitelyGeneratedModuleRep

▷ `IsGeneratorsOfFinitelyGeneratedModuleRep(rel)` (Representation)

Returns: `true` or `false`

The GAP representation of a finite set of generators of a finitely generated homalg module.
(It is a representation of the GAP representation `IsGeneratorsOfModuleRep` (6.1.4))

Code
```
DeclareRepresentation( "IsGeneratorsOfFinitelyGeneratedModuleRep",
        IsGeneratorsOfModuleRep,
        [ "generators", "relations_of_hullmodule" ] );
```

6.2 Generators: Constructors

6.3 Generators: Properties

6.3.1 IsReduced (for generators)

▷ `IsReduced(gen)` (property)

Returns: `true` or `false`

Check if the homalg set of generators *gen* is marked reduced.
(no method installed)

6.4 Generators: Attributes

6.4.1 ProcedureToReadjustGenerators

▷ `ProcedureToReadjustGenerators(gen)` (attribute)

Returns: a function

A function that takes the rows/columns of *gen* and returns an object (e.g. a matrix) that can be interpreted as a generator (this is important for modules of homomorphisms).

6.5 Generators: Operations and Functions

Chapter 7

Modules

A homalg module is a data structure for a finitely presented module. A presentation is given by a set of generators and a set of relations among these generators. The data structure for modules in homalg has two novel features:

- The data structure allows several presentations linked with so-called transition matrices. One of the presentations is marked as the default presentation, which is usually the last added one. A new presentation can always be added provided it is linked to the default presentation by a transition matrix. If needed, the user can reset the default presentation by choosing one of the other presentations saved in the data structure of the homalg module. Effectively, a module is then given by "all" its presentations (as "coordinates") together with isomorphisms between them (as "coordinate changes"). Being able to "change coordinates" makes the realization of a module in homalg *intrinsic* (or "coordinate free").

- To present a left/right module it suffices to take a matrix M and interpret its rows/columns as relations among n *abstract* generators, where n is the number of columns/rows of M. Only that these abstract generators are useless when it comes to specific modules like modules of homomorphisms, where one expects the generators to be maps between modules. For this reason a presentation of a module in homalg is not merely a matrix of relations, but together with a set of generators.

7.1 Modules: Category and Representations

7.1.1 IsHomalgModule

▷ `IsHomalgModule(M)` (Category)

Returns: `true` or `false`

The GAP category of homalg modules.

(It is a subcategory of the GAP categories `IsHomalgRingOrModule` and `IsHomalgStaticObject`.)

Code
```
DeclareCategory( "IsHomalgModule",
        IsHomalgRingOrModule and
        IsHomalgModuleOrMap and
        IsHomalgStaticObject );
```

7.1.2 IsFinitelyPresentedModuleOrSubmoduleRep

▷ `IsFinitelyPresentedModuleOrSubmoduleRep(M)` (Representation)

Returns: `true` or `false`

The GAP representation of finitley presented homalg modules or submodules.

(It is a representation of the GAP category `IsHomalgModule` (7.1.1), which is a subrepresentation of the GAP representations `IsStaticFinitelyPresentedObjectOrSubobjectRep`.)

Code
```
DeclareRepresentation( "IsFinitelyPresentedModuleOrSubmoduleRep",
        IsHomalgModule and
        IsStaticFinitelyPresentedObjectOrSubobjectRep,
        [ ] );
```

7.1.3 IsFinitelyPresentedModuleRep

▷ `IsFinitelyPresentedModuleRep(M)` (Representation)

Returns: `true` or `false`

The GAP representation of finitley presented homalg modules.

(It is a representation of the GAP category `IsHomalgModule` (7.1.1), which is a subrepresentation of the GAP representations `IsFinitelyPresentedModuleOrSubmoduleRep`, `IsStaticFinitelyPresentedObjectRep`, and `IsHomalgRingOrFinitelyPresentedModuleRep`.)

Code
```
DeclareRepresentation( "IsFinitelyPresentedModuleRep",
        IsFinitelyPresentedModuleOrSubmoduleRep and
        IsStaticFinitelyPresentedObjectRep and
        IsHomalgRingOrFinitelyPresentedModuleRep,
        [ "SetsOfGenerators", "SetsOfRelations",
          "PresentationMorphisms",
          "Resolutions",
          "TransitionMatrices",
          "PositionOfTheDefaultPresentation" ] );
```

7.1.4 IsFinitelyPresentedSubmoduleRep

▷ `IsFinitelyPresentedSubmoduleRep(M)` (Representation)

Returns: `true` or `false`

The GAP representation of finitley generated homalg submodules.

(It is a representation of the GAP category `IsHomalgModule` (7.1.1), which is a subrepresentation of the GAP representations `IsFinitelyPresentedModuleOrSubmoduleRep`, `IsStaticFinitelyPresentedSubobjectRep`, and `IsHomalgRingOrFinitelyPresentedModuleRep`.)

Code
```
DeclareRepresentation( "IsFinitelyPresentedSubmoduleRep",
        IsFinitelyPresentedModuleOrSubmoduleRep and
        IsStaticFinitelyPresentedSubobjectRep and
        IsHomalgRingOrFinitelyPresentedModuleRep,
        [ "map_having_subobject_as_its_image" ] );
```

7.2 Modules: Constructors

7.2.1 LeftPresentation (constructor for left modules)

▷ `LeftPresentation(mat)` (operation)

Returns: a homalg module

This constructor returns the finitely presented left module with relations given by the rows of the homalg matrix *mat*.

Example

```
gap> ZZ := HomalgRingOfIntegers( );;
gap> M := HomalgMatrix( "[ \
> 2, 3, 4, \
> 5, 6, 7  \
> ]", 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 3 generators>
gap> Display( M );
[ [  2,  3,  4 ],
  [  5,  6,  7 ] ]

Cokernel of the map

Z^(1x2) --> Z^(1x3),

currently represented by the above matrix
gap> ByASmallerPresentation( M );
<A rank 1 left module presented by 1 relation for 2 generators>
gap> Display( last );
Z/< 3 > + Z^(1 x 1)
```

7.2.2 RightPresentation (constructor for right modules)

▷ `RightPresentation(mat)` (operation)

Returns: a homalg module

This constructor returns the finitely presented right module with relations given by the columns of the homalg matrix *mat*.

Example

```
gap> ZZ := HomalgRingOfIntegers( );;
gap> M := HomalgMatrix( "[ \
> 2, 3, 4, \
> 5, 6, 7  \
> ]", 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
gap> M := RightPresentation( M );
<A right module on 2 generators satisfying 3 relations>
gap> ByASmallerPresentation( M );
<A cyclic torsion right module on a cyclic generator satisfying 1 relation>
gap> Display( last );
Z/< 3 >
```

7.2.3 HomalgFreeLeftModule (constructor for free left modules)

▷ HomalgFreeLeftModule(*r*, *R*) (operation)

Returns: a homalg module

This constructor returns a free left module of rank *r* over the homalg ring *R*.

Example
```
gap> ZZ := HomalgRingOfIntegers( );;
gap> F := HomalgFreeLeftModule( 1, ZZ );
<A free left module of rank 1 on a free generator>
gap> 1 * ZZ;
<The free left module of rank 1 on a free generator>
gap> F := HomalgFreeLeftModule( 2, ZZ );
<A free left module of rank 2 on free generators>
gap> 2 * ZZ;
<A free left module of rank 2 on free generators>
```

7.2.4 HomalgFreeRightModule (constructor for free right modules)

▷ HomalgFreeRightModule(*r*, *R*) (operation)

Returns: a homalg module

This constructor returns a free right module of rank *r* over the homalg ring *R*.

Example
```
gap> ZZ := HomalgRingOfIntegers( );;
gap> F := HomalgFreeRightModule( 1, ZZ );
<A free right module of rank 1 on a free generator>
gap> ZZ * 1;
<The free right module of rank 1 on a free generator>
gap> F := HomalgFreeRightModule( 2, ZZ );
<A free right module of rank 2 on free generators>
gap> ZZ * 2;
<A free right module of rank 2 on free generators>
```

7.2.5 HomalgZeroLeftModule (constructor for zero left modules)

▷ HomalgZeroLeftModule(*r*, *R*) (operation)

Returns: a homalg module

This constructor returns a zero left module of rank *r* over the homalg ring *R*.

Example
```
gap> ZZ := HomalgRingOfIntegers( );;
gap> F := HomalgZeroLeftModule( ZZ );
<A zero left module>
gap> 0 * ZZ;
<The zero left module>
```

7.2.6 HomalgZeroRightModule (constructor for zero right modules)

▷ HomalgZeroRightModule(*r*, *R*) (operation)

Returns: a homalg module

This constructor returns a zero right module of rank *r* over the homalg ring *R*.

Example
```
gap> ZZ := HomalgRingOfIntegers( );;
gap> F := HomalgZeroRightModule( ZZ );
<A zero right module>
gap> ZZ * 0;
<The zero right module>
```

7.2.7 * (transfer a module over a different ring)

▷ *(R, M) (operation)

▷ *(M, R) (operation)

Returns: a homalg module

Transfers the S-module `M` over the homalg ring `R`. This works only in three cases:

1. S is a subring of `R`.
2. `R` is a residue class ring of S constructed using `/`.
3. `R` is a subring of S and the entries of the current matrix of S-relations of `M` lie in `R`.

CAUTION: So it is not suited for general base change.

Example
```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> Display( ZZ );
<An internal ring>
gap> Z4 := ZZ / 4;
Z/( 4 )
gap> Display( Z4 );
<A residue class ring>
gap> M := HomalgDiagonalMatrix( [ 2 .. 4 ], ZZ );
<An unevaluated diagonal 3 x 3 matrix over an internal ring>
gap> M := LeftPresentation( M );
<A torsion left module presented by 3 relations for 3 generators>
gap> Display( M );
Z/< 2 > + Z/< 3 > + Z/< 4 >
gap> M;
<A torsion left module presented by 3 relations for 3 generators>
gap> N := Z4 * M; ## or N := M * Z4;
<A non-torsion left module presented by 2 relations for 3 generators>
gap> ByASmallerPresentation( N );
<A non-torsion left module presented by 1 relation for 2 generators>
gap> Display( N );
Z/( 4 )/< |[ 2 ]| > + Z/( 4 )^(1 x 1)
gap> N;
<A non-torsion left module presented by 1 relation for 2 generators>
```

Example
```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> M := HomalgMatrix( "[ \
> 2, 3, 4, \
> 5, 6, 7  \
```

```
> ]", 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 3 generators>
gap> Z4 := ZZ / 4;
Z/( 4 )
gap> Display( Z4 );
<A residue class ring>
gap> M4 := Z4 * M;
<A non-torsion left module presented by 2 relations for 3 generators>
gap> Display( M4 );
[ [  2,  3,  4 ],
  [  5,  6,  7 ] ]

modulo [ 4 ]

Cokernel of the map

Z/( 4 )^(1x2) --> Z/( 4 )^(1x3),

currently represented by the above matrix
gap> d := Resolution( 2, M4 );
<A right acyclic complex containing 2 morphisms of left modules at degrees
[ 0 .. 2 ]>
gap> dd := Hom( d, Z4 );
<A cocomplex containing 2 morphisms of right modules at degrees [ 0 .. 2 ]>
gap> DD := Resolution( 2, dd );
<A cocomplex containing 2 morphisms of right complexes at degrees [ 0 .. 2 ]>
gap> D := Hom( DD, Z4 );
<A complex containing 2 morphisms of left cocomplexes at degrees [ 0 .. 2 ]>
gap> C := ZZ * D;
<A "complex" containing 2 morphisms of left cocomplexes at degrees [ 0 .. 2 ]>
gap> LowestDegreeObject( C );
<A "cocomplex" containing 2 morphisms of left modules at degrees [ 0 .. 2 ]>
gap> Display( last );
-------------------------
at cohomology degree: 2
0
------------^------------
(an empty 1 x 0 matrix)

the map is currently represented by the above 1 x 0 matrix
-------------------------
at cohomology degree: 1
Z/< 4 >
------------^------------
[ [  0 ],
  [  1 ],
  [  2 ],
  [  1 ] ]

the map is currently represented by the above 4 x 1 matrix
```

```
-------------------------
at cohomology degree: 0
Z/< 4 > + Z/< 4 > + Z/< 4 > + Z/< 4 >
-------------------------
```

7.2.8 Subobject (constructor for submodules using matrices)

▷ `Subobject(mat, M)` (operation)

Returns: a homalg submodule

This constructor returns the finitely generated left/right submodule of the homalg module *M* with generators given by the rows/columns of the homalg matrix *mat*.

7.2.9 Subobject (constructor for submodules using a list of ring elements)

▷ `Subobject(gens, M)` (operation)

Returns: a homalg submodule

This constructor returns the finitely generated left/right submodule of the homalg cyclic left/right module *M* with generators given by the entries of the list *gens*.

7.2.10 LeftSubmodule (constructor for left submodules)

▷ `LeftSubmodule(mat)` (operation)

Returns: a homalg submodule

This constructor returns the finitely generated left submodule with generators given by the rows of the homalg matrix *mat*.

Code
```
InstallMethod( LeftSubmodule,
        "constructor for homalg submodules",
        [ IsHomalgMatrix ],

  function( gen )
    local R;

    R := HomalgRing( gen );

    return Subobject( gen, NrColumns( gen ) * R );

end );
```

Example
```
gap> Z4 := HomalgRingOfIntegers( ) / 4;
Z/( 4 )
gap> I := HomalgMatrix( "[ 2 ]", 1, 1, Z4 );
<A 1 x 1 matrix over a residue class ring>
gap> I := LeftSubmodule( I );
<A principal torsion-free (left) ideal given by a cyclic generator>
gap> IsFree( I );
false
gap> I;
<A principal reflexive non-projective (left) ideal given by a cyclic generator\
>
```

7.2.11 RightSubmodule (constructor for right submodules)

▷ RightSubmodule(*mat*) (operation)

Returns: a homalg submodule

This constructor returns the finitely generated right submodule with generators given by the columns of the homalg matrix *mat*.

Code
```
InstallMethod( RightSubmodule,
        "constructor for homalg submodules",
        [ IsHomalgMatrix ],

  function( gen )
    local R;

    R := HomalgRing( gen );

    return Subobject( gen, R * NrRows( gen ) );

end );
```

Example
```
gap> Z4 := HomalgRingOfIntegers( ) / 4;
Z/( 4 )
gap> I := HomalgMatrix( "[ 2 ]", 1, 1, Z4 );
<A 1 x 1 matrix over a residue class ring>
gap> I := RightSubmodule( I );
<A principal torsion-free (right) ideal given by a cyclic generator>
gap> IsFree( I );
false
gap> I;
<A principal reflexive non-projective (right) ideal given by a cyclic generato\
r>
```

7.3 Modules: Properties

7.3.1 IsCyclic

▷ IsCyclic(*M*) (property)

Returns: `true` or `false`

Check if the homalg module *M* is cyclic.

7.3.2 IsHolonomic

▷ IsHolonomic(*M*) (property)

Returns: `true` or `false`

Check if the homalg module *M* is holonomic.

7.3.3 IsReduced (for modules)

▷ IsReduced(*M*) (property)

Returns: `true` or `false`

Check if the homalg module M is reduced.

7.3.4 IsPrimeIdeal

▷ `IsPrimeIdeal(J)` (property)

Returns: `true` or `false`

Check if the homalg submodule J is a prime ideal. The ring has to be commutative.

(no method installed)

7.3.5 IsPrimeModule (for modules)

▷ `IsPrimeModule(M)` (property)

Returns: `true` or `false`

Check if the homalg module M is prime.

For more properties see the corresponding section (**homalg: Objects: Properties**)) in the documentation of the homalg package.

7.4 Modules: Attributes

7.4.1 ResidueClassRing

▷ `ResidueClassRing(J)` (attribute)

Returns: a homalg ring

In case J was defined as a (left/right) ideal of the ring R the residue class ring R/J is returned.

7.4.2 PrimaryDecomposition

▷ `PrimaryDecomposition(J)` (attribute)

Returns: a list

The primary decomposition of the ideal J. The ring has to be commutative.

(no method installed)

7.4.3 RadicalDecomposition

▷ `RadicalDecomposition(J)` (attribute)

Returns: a list

The prime decomposition of the radical of the ideal J. The ring has to be commutative.

(no method installed)

7.4.4 ModuleOfKaehlerDifferentials

▷ `ModuleOfKaehlerDifferentials(R)` (attribute)

Returns: a homalg module

The module of Kaehler differentials of the (residue class ring) R.

(method installed in package GradedModules)

7.4.5 RadicalSubobject

▷ `RadicalSubobject(M)` (property)

Returns: a function

M is a homalg module.

7.4.6 SymmetricAlgebra

▷ `SymmetricAlgebra(M)` (attribute)

Returns: a homalg ring

The symmetric algebra of the module *M*.

7.4.7 ExteriorAlgebra

▷ `ExteriorAlgebra(M)` (attribute)

Returns: a homalg ring

The exterior algebra of the module *M*.

7.4.8 ElementaryDivisors

▷ `ElementaryDivisors(M)` (attribute)

Returns: a list of ring elements

The list of elementary divisors of the homalg module *M*, in case they exist.
(no method installed)

7.4.9 FittingIdeal

▷ `FittingIdeal(M)` (attribute)

Returns: a list

The Fitting ideal of *M*.

7.4.10 NonFlatLocus

▷ `NonFlatLocus(M)` (attribute)

Returns: a list

The non flat locus of *M*.

7.4.11 LargestMinimalNumberOfLocalGenerators

▷ `LargestMinimalNumberOfLocalGenerators(M)` (attribute)

Returns: a nonnegative integer

The minimal number of *local* generators of the module *M*.

7.4.12 CoefficientsOfUnreducedNumeratorOfHilbertPoincareSeries

▷ `CoefficientsOfUnreducedNumeratorOfHilbertPoincareSeries(M)` (attribute)

Returns: a list of integers

M is a homalg module.

7.4.13 CoefficientsOfNumeratorOfHilbertPoincareSeries

▷ `CoefficientsOfNumeratorOfHilbertPoincareSeries(M)` (attribute)

Returns: a list of integers

M is a homalg module.

7.4.14 UnreducedNumeratorOfHilbertPoincareSeries

▷ `UnreducedNumeratorOfHilbertPoincareSeries(M)` (attribute)

Returns: a univariate polynomial with rational coefficients

M is a homalg module.

7.4.15 NumeratorOfHilbertPoincareSeries

▷ `NumeratorOfHilbertPoincareSeries(M)` (attribute)

Returns: a univariate polynomial with rational coefficients

M is a homalg module.

7.4.16 HilbertPoincareSeries

▷ `HilbertPoincareSeries(M)` (attribute)

Returns: a univariate rational function with rational coefficients

M is a homalg module.

7.4.17 AffineDegree

▷ `AffineDegree(M)` (attribute)

Returns: a nonnegative integer

M is a homalg module.

7.4.18 DataOfHilbertFunction

▷ `DataOfHilbertFunction(M)` (property)

Returns: a function

M is a homalg module.

7.4.19 HilbertFunction

▷ `HilbertFunction(M)` (property)

Returns: a function

M is a homalg module.

7.4.20 IndexOfRegularity

▷ `IndexOfRegularity(M)` (property)

Returns: a function

M is a homalg module.

For more attributes see the corresponding section (**homalg: Objects: Attributes**)) in the documentation of the homalg package.

7.5 Modules: Operations and Functions

7.5.1 HomalgRing (for modules)

▷ `HomalgRing(M)` (operation)

Returns: a homalg ring

The homalg ring of the homalg module `M`.

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> M := ZZ * 4;
<A free right module of rank 4 on free generators>
gap> R := HomalgRing( M );
Z
gap> IsIdenticalObj( R, ZZ );
true
```

7.5.2 ByASmallerPresentation (for modules)

▷ `ByASmallerPresentation(M)` (method)

Returns: a homalg module

Use different strategies to reduce the presentation of the given homalg module `M`. This method performs side effects on its argument `M` and returns it.

Example

```
gap> ZZ := HomalgRingOfIntegers( );;
gap> M := HomalgMatrix( "[ \
> 2, 3, 4, \
> 5, 6, 7  \
> ]", 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 3 generators>
gap> Display( M );
[ [  2,  3,  4 ],
  [  5,  6,  7 ] ]

Cokernel of the map

Z^(1x2) --> Z^(1x3),

currently represented by the above matrix
gap> ByASmallerPresentation( M );
<A rank 1 left module presented by 1 relation for 2 generators>
gap> Display( last );
Z/< 3 > + Z^(1 x 1)
gap> SetsOfGenerators( M );
<A set containing 2 sets of generators of a homalg module>
gap> SetsOfRelations( M );
<A set containing 2 sets of relations of a homalg module>
gap> M;
<A rank 1 left module presented by 1 relation for 2 generators>
gap> SetPositionOfTheDefaultPresentation( M, 1 );
```

```
gap> M;
<A rank 1 left module presented by 2 relations for 3 generators>
```

7.5.3 * (constructor for ideal multiples)

▷ `\*(J, M)` (operation)

Returns: a homalg submodule

Compute the submodule JM (resp. MJ) of the given left (resp. right) R-module M, where J is a left (resp. right) ideal in R.

7.5.4 SubobjectQuotient (for submodules)

▷ `SubobjectQuotient(K, J)` (operation)

Returns: a homalg ideal

Compute the submodule quotient ideal $K : J$ of the submodules K and J of a common R-module M.

Chapter 8

Maps

A homalg map is a data structures for maps (module homomorphisms) between finitely generated modules. Each map in homalg knows its source (→ `Source` (**homalg: Source**)) and its target (→ `Range` (**homalg: Range**)). A map is represented by a homalg matrix relative to the current set of generators of the source resp. target homalg module. As with modules (→ Chapter 7), maps in homalg are realized in an intrinsic manner: If the presentations of the source or/and target module are altered after the map was constructed, a new adapted representation matrix of the map is automatically computed whenever needed. For this the internal transition matrices of the modules are used. homalg uses the so-called *associative* convention for maps. This means that maps of left modules are applied from the right, whereas maps of right modules from the left.

8.1 Maps: Categories and Representations

8.1.1 IsHomalgMap

▷ `IsHomalgMap(`*phi*`)` (Category)

Returns: `true` or `false`

The GAP category of homalg maps.

(It is a subcategory of the GAP categories `IsHomalgModuleOrMap` and `IsHomalgStaticMorphism`.)

Code
```
DeclareCategory( "IsHomalgMap",
        IsHomalgModuleOrMap and
        IsHomalgStaticMorphism );
```

8.1.2 IsHomalgSelfMap

▷ `IsHomalgSelfMap(`*phi*`)` (Category)

Returns: `true` or `false`

The GAP category of homalg self-maps.

(It is a subcategory of the GAP categories `IsHomalgMap` and `IsHomalgEndomorphism`.)

Code
```
DeclareCategory( "IsHomalgSelfMap",
        IsHomalgMap and
        IsHomalgEndomorphism );
```

8.1.3 IsMapOfFinitelyGeneratedModulesRep

▷ `IsMapOfFinitelyGeneratedModulesRep(phi)` (Representation)

Returns: `true` or `false`

The GAP representation of maps between finitley generated homalg modules.

(It is a representation of the GAP category `IsHomalgChainMorphism` (**homalg: IsHomalgChainMorphism**), which is a subrepresentation of the GAP representation `IsStaticMorphismOfFinitelyGeneratedObjectsRep`.)

8.2 Maps: Constructors

8.2.1 HomalgMap (constructor for maps)

▷ `HomalgMap(mat, M, N)` (function)

▷ `HomalgMap(mat[, string])` (function)

Returns: a homalg map

This constructor returns a map (homomorphism) of finitely presented modules. It is represented by the homalg matrix *mat* relative to the current set of generators of the source homalg module *M* and target module *N* (→ 7.2). Unless the source module is free *and* given on free generators the returned map will cautiously be indicated using parenthesis: “homomorphism”. To verify if the result is indeed a well defined map use `IsMorphism` (**homalg: IsMorphism**). If the presentations of the source or/and target module are altered after the map was constructed, a new adapted representation matrix of the map is automatically computed whenever needed. For this the internal transition matrices of the modules are used. If source and target are identical objects, and only then, the map is created as a selfmap (endomorphism). homalg uses the so-called *associative* convention for maps. This means that maps of left modules are applied from the right, whereas maps of right modules from the left.

Example

```
gap> ZZ := HomalgRingOfIntegers( );;
gap> M := HomalgMatrix( "[ 2, 3, 4,   5, 6, 7 ]", 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 3 generators>
gap> N := HomalgMatrix( "[ 2, 3, 4, 5,   6, 7, 8, 9 ]", 2, 4, ZZ );
<A 2 x 4 matrix over an internal ring>
gap> N := LeftPresentation( N );
<A non-torsion left module presented by 2 relations for 4 generators>
gap> mat := HomalgMatrix( "[ \
> 1, 0, -2, -4, \
> 0, 1,  4,  7, \
> 1, 0, -2, -4  \
> ]", 3, 4, ZZ );
<A 3 x 4 matrix over an internal ring>
gap> phi := HomalgMap( mat, M, N );
<A "homomorphism" of left modules>
gap> IsMorphism( phi );
true
gap> phi;
<A homomorphism of left modules>
gap> Display( phi );
[ [   1,   0,  -2,  -4 ],
```

```
  [   0,   1,   4,   7 ],
  [   1,   0,  -2,  -4 ] ]

the map is currently represented by the above 3 x 4 matrix
gap> ByASmallerPresentation( M );
<A rank 1 left module presented by 1 relation for 2 generators>
gap> Display( last );
Z/< 3 > + Z^(1 x 1)
gap> Display( phi );
[ [   2,   1,   0,  -1 ],
  [   1,   0,  -2,  -4 ] ]

the map is currently represented by the above 2 x 4 matrix
gap> ByASmallerPresentation( N );
<A rank 2 left module presented by 1 relation for 3 generators>
gap> Display( N );
Z/< 4 > + Z^(1 x 2)
gap> Display( phi );
[ [  -8,   0,   0 ],
  [  -3,  -1,  -2 ] ]

the map is currently represented by the above 2 x 3 matrix
gap> ByASmallerPresentation( phi );
<A non-zero homomorphism of left modules>
gap> Display( phi );
[ [   0,   0,   0 ],
  [   1,  -1,  -2 ] ]

the map is currently represented by the above 2 x 3 matrix
```

To construct a map with source being a not yet specified free module

Example

```
gap> N;
<A rank 2 left module presented by 1 relation for 3 generators>
gap> SetPositionOfTheDefaultSetOfGenerators( N, 1 );
gap> N;
<A rank 2 left module presented by 2 relations for 4 generators>
gap> psi := HomalgMap( mat, "free", N );
<A homomorphism of left modules>
gap> Source( psi );
<A free left module of rank 3 on free generators>
```

To construct a map between not yet specified free left modules

Example

```
gap> chi := HomalgMap( mat ); ## or chi := HomalgMap( mat, "l" );
<A homomorphism of left modules>
gap> Source( chi );
<A free left module of rank 3 on free generators>
gap> Range( chi );
<A free left module of rank 4 on free generators>
```

To construct a map between not yet specified free right modules

Example
```
gap> kappa := HomalgMap( mat, "r" );
<A homomorphism of right modules>
gap> Source( kappa );
<A free right module of rank 4 on free generators>
gap> Range( kappa );
<A free right module of rank 3 on free generators>
```

8.2.2 HomalgZeroMap (constructor for zero maps)

▷ HomalgZeroMap(*M*, *N*) (function)

Returns: a homalg map

The constructor returns the zero map between the source homalg module *M* and the target homalg module *N*.

Example
```
gap> ZZ := HomalgRingOfIntegers( );;
gap> M := HomalgMatrix( "[ 2, 3, 4,   5, 6, 7 ]", 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 3 generators>
gap> N := HomalgMatrix( "[ 2, 3, 4, 5,   6, 7, 8, 9 ]", 2, 4, ZZ );
<A 2 x 4 matrix over an internal ring>
gap> N := LeftPresentation( N );
<A non-torsion left module presented by 2 relations for 4 generators>
gap> HomalgZeroMap( M, N );
<The zero morphism of left modules>
```

8.2.3 HomalgIdentityMap (constructor for identity maps)

▷ HomalgIdentityMap(*M*, *N*) (function)

Returns: a homalg map

The constructor returns the identity map of the homalg module *M*.

Example
```
gap> ZZ := HomalgRingOfIntegers( );;
gap> M := HomalgMatrix( "[ 2, 3, 4,   5, 6, 7 ]", 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 3 generators>
gap> HomalgIdentityMap( M );
<The identity morphism of a non-zero left module>
```

8.3 Maps: Properties

8.4 Maps: Attributes

8.5 Maps: Operations and Functions

8.5.1 HomalgRing

▷ `HomalgRing(phi)` (operation)

Returns: a homalg ring

The homalg ring of the homalg map *phi*.

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> phi := HomalgIdentityMap( 2 * ZZ );
<The identity morphism of a non-zero left module>
gap> R := HomalgRing( phi );
Z
gap> IsIdenticalObj( R, ZZ );
true
```

8.5.2 PreInverse

▷ `PreInverse(phi)` (operation)

Returns: a homalg map, `false`, or `fail`

Compute a pre-inverse of the morphism *phi* in case one exists. For a pre-inverse to exist *phi* must be an epimorphism. For *commutative* rings homalg has an algorithm installed which decides the existence and returns a pre-inverse in case one exists. If a pre-inverse does not exist then `false` is returned. The algorithm finds a particular solution of a two-side inhomogeneous linear system over R :=`HomalgRing`(*phi*). For *non*commutative rings a heuristic method is installed. If it finds a pre-inverse it returns it, otherwise it returns `fail` (→ 1.1.3). The operation `PreInverse` is used to install a method for the property `IsSplitEpimorphism` (**homalg: IsSplitEpimorphism**).

`PreInverse` checks if it can decide the projectivity of `Range`(*phi*).

Chapter 9

Module Elements

An element of a module M is internally represented by a module map from the (distinguished) rank 1 free module to the module M. In particular, the data structure for module elements automatically profits from the intrinsic realization of morphisms in the homalg project.

9.1 Module Elements: Category and Representations

9.1.1 IsHomalgElement

▷ `IsHomalgElement(M)` (Category)

Returns: `true` or `false`

The GAP category of module elements.

9.1.2 IsElementOfAModuleGivenByAMorphismRep

▷ `IsElementOfAModuleGivenByAMorphismRep(M)` (Representation)

Returns: `true` or `false`

The GAP representation of elements of modules.

(It is a subrepresentation of `IsElementOfAnObjectGivenByAMorphismRep` (**homalg: IsElementOfAnObjectGivenByAMorphismRep**).)

9.2 Module Elements: Constructors

9.3 Module Elements: Properties

9.3.1 IsElementOfIntegers

▷ `IsElementOfIntegers(m)` (property)

Returns: `true` or `false`

Check if the *m* is an element of the integers viewed as a module over itself.

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> a := HomalgElement( HomalgMap( "[[2]]", 1 * ZZ, 1 * ZZ ) );
2
```

```
gap> IsElementOfIntegers( a );
true
gap> Z4 := ZZ / 4;
Z/( 4 )
gap> b := HomalgElement( HomalgMap( "[[-1]]", 1 * Z4, 1 * Z4 ) );
|[ 3 ]|
gap> IsElementOfIntegers( b );
false
```

9.4 Module Elements: Attributes

9.5 Module Elements: Operations and Functions

9.5.1 HomalgRing (for module elements)

▷ `HomalgRing(m)` (operation)

Returns: a homalg ring

The homalg ring of the homalg module element *m*.

Example
```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> a := HomalgElement( HomalgMap( "[[2]]", 1 * ZZ, 1 * ZZ ) );
2
gap> IsIdenticalObj( ZZ, HomalgRing( a ) );
true
```

Chapter 10

Functors

10.1 Functors: Category and Representations

10.2 Functors: Constructors

10.3 Functors: Attributes

10.4 Basic Functors

10.4.1 functor_Cokernel

▷ `functor_Cokernel` (global variable)

The functor that associates to a map its cokernel.

Code
```
InstallValue( functor_Cokernel_for_fp_modules,
        CreateHomalgFunctor(
                [ "name", "Cokernel" ],
                [ "category", HOMALG_MODULES.category ],
                [ "operation", "Cokernel" ],
                [ "natural_transformation", "CokernelEpi" ],
                [ "special", true ],
                [ "number_of_arguments", 1 ],
                [ "1", [ [ "covariant" ],
                        [ IsMapOfFinitelyGeneratedModulesRep,
                          [ IsHomalgChainMorphism, IsImageSquare ] ] ] ],
                [ "OnObjects", _Functor_Cokernel_OnModules ]
                )
        );
```

10.4.2 Cokernel

▷ `Cokernel(`*phi*`)` (operation)

The following example also makes use of the natural transformation `CokernelEpi`.

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> M := HomalgMatrix( "[ 2, 3, 4,   5, 6, 7 ]", 2, 3, ZZ );;
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 3 generators>
gap> N := HomalgMatrix( "[ 2, 3, 4, 5,   6, 7, 8, 9 ]", 2, 4, ZZ );;
gap> N := LeftPresentation( N );
<A non-torsion left module presented by 2 relations for 4 generators>
gap> mat := HomalgMatrix( "[ \
> 1, 0, -3, -6, \
> 0, 1,  6, 11, \
> 1, 0, -3, -6  \
> ]", 3, 4, ZZ );;
gap> phi := HomalgMap( mat, M, N );;
gap> IsMorphism( phi );
true
gap> phi;
<A homomorphism of left modules>
gap> coker := Cokernel( phi );
<A left module presented by 5 relations for 4 generators>
gap> ByASmallerPresentation( coker );
<A rank 1 left module presented by 1 relation for 2 generators>
gap> Display( coker );
Z/< 8 > + Z^(1 x 1)
gap> nu := CokernelEpi( phi );
<An epimorphism of left modules>
gap> Display( nu );
[ [  -5,   0 ],
  [  -6,   1 ],
  [   1,  -2 ],
  [   0,   1 ] ]

the map is currently represented by the above 4 x 2 matrix
gap> DefectOfExactness( phi, nu );
<A zero left module>
gap> ByASmallerPresentation( nu );
<A non-zero epimorphism of left modules>
gap> Display( nu );
[ [   2,   0 ],
  [   1,  -2 ],
  [   0,   1 ] ]

the map is currently represented by the above 3 x 2 matrix
gap> PreInverse( nu );
false
```

10.4.3 functor_ImageObject

▷ `functor_ImageObject` (global variable)

The functor that associates to a map its image.

```
InstallValue( functor_ImageObject_for_fp_modules,
        CreateHomalgFunctor(
                [ "name", "ImageObject for modules" ],
                [ "category", HOMALG_MODULES.category ],
                [ "operation", "ImageObject" ],
                [ "natural_transformation", "ImageObjectEmb" ],
                [ "number_of_arguments", 1 ],
                [ "1", [ [ "covariant" ],
                        [ IsMapOfFinitelyGeneratedModulesRep and
                          AdmissibleInputForHomalgFunctors ] ] ],
                [ "OnObjects", _Functor_ImageObject_OnModules ]
                )
        );
```

10.4.4 ImageObject

▷ ImageObject(*phi*) (operation)

The following example also makes use of the natural transformations `ImageObjectEpi` and `ImageObjectEmb`.

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> M := HomalgMatrix( "[ 2, 3, 4,   5, 6, 7 ]", 2, 3, ZZ );;
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 3 generators>
gap> N := HomalgMatrix( "[ 2, 3, 4, 5,   6, 7, 8, 9 ]", 2, 4, ZZ );;
gap> N := LeftPresentation( N );
<A non-torsion left module presented by 2 relations for 4 generators>
gap> mat := HomalgMatrix( "[ \
> 1, 0, -3, -6, \
> 0, 1,  6, 11, \
> 1, 0, -3, -6  \
> ]", 3, 4, ZZ );;
gap> phi := HomalgMap( mat, M, N );;
gap> IsMorphism( phi );
true
gap> phi;
<A homomorphism of left modules>
gap> im := ImageObject( phi );
<A left module presented by yet unknown relations for 3 generators>
gap> ByASmallerPresentation( im );
<A free left module of rank 1 on a free generator>
gap> pi := ImageObjectEpi( phi );
<A non-zero split epimorphism of left modules>
gap> epsilon := ImageObjectEmb( phi );
<A monomorphism of left modules>
gap> phi = pi * epsilon;
true
```

10.4.5 Kernel (for maps)

▷ `Kernel(phi)` (operation)

The following example also makes use of the natural transformation `KernelEmb`.

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> M := HomalgMatrix( "[ 2, 3, 4,   5, 6, 7 ]", 2, 3, ZZ );;
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 3 generators>
gap> N := HomalgMatrix( "[ 2, 3, 4, 5,   6, 7, 8, 9 ]", 2, 4, ZZ );;
gap> N := LeftPresentation( N );
<A non-torsion left module presented by 2 relations for 4 generators>
gap> mat := HomalgMatrix( "[ \
> 1, 0, -3, -6, \
> 0, 1,  6, 11, \
> 1, 0, -3, -6  \
> ]", 3, 4, ZZ );;
gap> phi := HomalgMap( mat, M, N );;
gap> IsMorphism( phi );
true
gap> phi;
<A homomorphism of left modules>
gap> ker := Kernel( phi );
<A cyclic left module presented by yet unknown relations for a cyclic generato\
r>
gap> Display( ker );
Z/< -3 >
gap> ByASmallerPresentation( last );
<A cyclic torsion left module presented by 1 relation for a cyclic generator>
gap> Display( ker );
Z/< 3 >
gap> iota := KernelEmb( phi );
<A monomorphism of left modules>
gap> Display( iota );
[ [  0,  2,  4 ] ]

the map is currently represented by the above 1 x 3 matrix
gap> DefectOfExactness( iota, phi );
<A zero left module>
gap> ByASmallerPresentation( iota );
<A non-zero monomorphism of left modules>
gap> Display( iota );
[ [  2,  0 ] ]

the map is currently represented by the above 1 x 2 matrix
gap> PostInverse( iota );
fail
```

10.4.6 DefectOfExactness

▷ DefectOfExactness(*phi*, *psi*) (operation)

We follow the associative convention for applying maps. For left modules *phi* is applied first and from the right. For right modules *psi* is applied first and from the left.

The following example also makes use of the natural transformation KernelEmb.

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> M := HomalgMatrix( "[ 2, 3, 4, 0,   5, 6, 7, 0 ]", 2, 4, ZZ );;
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 4 generators>
gap> N := HomalgMatrix( "[ 2, 3, 4, 5,   6, 7, 8, 9 ]", 2, 4, ZZ );;
gap> N := LeftPresentation( N );
<A non-torsion left module presented by 2 relations for 4 generators>
gap> mat := HomalgMatrix( "[ \
> 1, 3,  3,  3, \
> 0, 3, 10, 17, \
> 1, 3,  3,  3, \
> 0, 0,  0,  0  \
> ]", 4, 4, ZZ );;
gap> phi := HomalgMap( mat, M, N );;
gap> IsMorphism( phi );
true
gap> phi;
<A homomorphism of left modules>
gap> iota := KernelEmb( phi );
<A monomorphism of left modules>
gap> DefectOfExactness( iota, phi );
<A zero left module>
gap> hom_iota := Hom( iota ); ## a shorthand for Hom( iota, ZZ );
<A homomorphism of right modules>
gap> hom_phi := Hom( phi ); ## a shorthand for Hom( phi, ZZ );
<A homomorphism of right modules>
gap> DefectOfExactness( hom_iota, hom_phi );
<A cyclic right module on a cyclic generator satisfying yet unknown relations>
gap> ByASmallerPresentation( last );
<A cyclic torsion right module on a cyclic generator satisfying 1 relation>
gap> Display( last );
Z/< 2 >
```

10.4.7 Functor_Hom

▷ Functor_Hom (global variable)

The bifunctor Hom.

Code

```
InstallValue( Functor_Hom_for_fp_modules,
        CreateHomalgFunctor(
                [ "name", "Hom" ],
                [ "category", HOMALG_MODULES.category ],
```

```
                  [ "operation", "Hom" ],
                  [ "number_of_arguments", 2 ],
                  [ "1", [ [ "contravariant", "right adjoint", "distinguished" ] ] ],
                  [ "2", [ [ "covariant", "left exact" ] ] ],
                  [ "OnObjects", _Functor_Hom_OnModules ],
                  [ "OnMorphisms", _Functor_Hom_OnMaps ],
                  [ "MorphismConstructor", HOMALG_MODULES.category.MorphismConstructor ]
                  )
        );
```

10.4.8 Hom

▷ Hom(*o1*, *o2*) (operation)

o1 resp. *o2* could be a module, a map, a complex (of modules or of again of complexes), or a chain morphism.

Each generator of a module of homomorphisms is displayed as a matrix of appropriate dimensions.

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> M := HomalgMatrix( "[ 2, 3, 4,   5, 6, 7 ]", 2, 3, ZZ );;
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 3 generators>
gap> N := HomalgMatrix( "[ 2, 3, 4, 5,   6, 7, 8, 9 ]", 2, 4, ZZ );;
gap> N := LeftPresentation( N );
<A non-torsion left module presented by 2 relations for 4 generators>
gap> mat := HomalgMatrix( "[ \
> 1, 0, -3, -6, \
> 0, 1,  6, 11, \
> 1, 0, -3, -6  \
> ]", 3, 4, ZZ );;
gap> phi := HomalgMap( mat, M, N );;
gap> IsMorphism( phi );
true
gap> phi;
<A homomorphism of left modules>
gap> psi := Hom( phi, M );
<A homomorphism of right modules>
gap> ByASmallerPresentation( psi );
<A non-zero homomorphism of right modules>
gap> Display( psi );
[ [   1,   1,   0,   1 ],
  [   2,   2,   0,   0 ],
  [   0,   0,   6,  10 ] ]

the map is currently represented by the above 3 x 4 matrix
gap> homNM := Source( psi );
<A rank 2 right module on 4 generators satisfying 2 relations>
gap> IsIdenticalObj( homNM, Hom( N, M ) ); ## the caching at work
true
gap> homMM := Range( psi );
```

```
<A rank 1 right module on 3 generators satisfying 2 relations>
gap> IsIdenticalObj( homMM, Hom( M, M ) ); ## the caching at work
true
gap> Display( homNM );
Z/< 3 > + Z/< 3 > + Z^(2 x 1)
gap> Display( homMM );
Z/< 3 > + Z/< 3 > + Z^(1 x 1)
gap> IsMonomorphism( psi );
false
gap> IsEpimorphism( psi );
false
gap> GeneratorsOfModule( homMM );
<A set of 3 generators of a homalg right module>
gap> Display( last );
[ [  0,  0,  0 ],
  [  0,  1,  2 ],
  [  0,  0,  0 ] ]

the map is currently represented by the above 3 x 3 matrix

[ [  0,  2,  4 ],
  [  0,  0,  0 ],
  [  0,  2,  4 ] ]

the map is currently represented by the above 3 x 3 matrix

[ [   0,   1,   3 ],
  [   0,   0,  -2 ],
  [   0,   1,   3 ] ]

the map is currently represented by the above 3 x 3 matrix

a set of 3 generators given by the the above matrices
gap> GeneratorsOfModule( homNM );
<A set of 4 generators of a homalg right module>
gap> Display( last );
[ [  0,  1,  2 ],
  [  0,  1,  2 ],
  [  0,  1,  2 ],
  [  0,  0,  0 ] ]

the map is currently represented by the above 4 x 3 matrix

[ [  0,  1,  2 ],
  [  0,  0,  0 ],
  [  0,  0,  0 ],
  [  0,  2,  4 ] ]

the map is currently represented by the above 4 x 3 matrix

[ [   0,   0,  -3 ],
  [   0,   0,   7 ],
```

```
  [   0,    0,  -5 ],
  [   0,    0,   1 ] ]

the map is currently represented by the above 4 x 3 matrix

[ [   0,    1,  -3 ],
  [   0,    0,  12 ],
  [   0,    0,  -9 ],
  [   0,    2,   6 ] ]

the map is currently represented by the above 4 x 3 matrix

a set of 4 generators given by the the above matrices
```

If for example the source N gets a new presentation, you will see the effect on the generators:

Example

```
gap> ByASmallerPresentation( N );
<A rank 2 left module presented by 1 relation for 3 generators>
gap> GeneratorsOfModule( homNM );
<A set of 4 generators of a homalg right module>
gap> Display( last );
[ [  0,  3,  6 ],
  [  0,  1,  2 ],
  [  0,  0,  0 ] ]

the map is currently represented by the above 3 x 3 matrix

[ [   0,    9,  18 ],
  [   0,    0,   0 ],
  [   0,    2,   4 ] ]

the map is currently represented by the above 3 x 3 matrix

[ [   0,    0,   0 ],
  [   0,    0,  -5 ],
  [   0,    0,   1 ] ]

the map is currently represented by the above 3 x 3 matrix

[ [   0,    9,  18 ],
  [   0,    0,  -9 ],
  [   0,    2,   6 ] ]

the map is currently represented by the above 3 x 3 matrix

a set of 4 generators given by the the above matrices
```

Now we compute a certain natural filtration on Hom(M,M):

Example

```
gap> dM := Resolution( M );
<A non-zero right acyclic complex containing a single morphism of left modules\
 at degrees [ 0 .. 1 ]>
```

```
gap> hMM := Hom( dM, dM );
<A non-zero acyclic cocomplex containing a single morphism of right complexes \
at degrees [ 0 .. 1 ]>
gap> BMM := HomalgBicomplex( hMM );
<A non-zero bicocomplex containing right modules at bidegrees [ 0 .. 1 ]x
[ -1 .. 0 ]>
gap> II_E := SecondSpectralSequenceWithFiltration( BMM );
<A stable cohomological spectral sequence with sheets at levels
[ 0 .. 2 ] each consisting of right modules at bidegrees [ -1 .. 0 ]x
[ 0 .. 1 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a cohomological spectral sequence at bidegrees
[ [ 0 .. 1 ], [ -1 .. 0 ] ]
---------
Level 0:

 * *
 * *
---------
Level 1:

 * *
 . .
---------
Level 2:

 s s
 . .

Now the spectral sequence of the bicomplex:

a cohomological spectral sequence at bidegrees
[ [ -1 .. 0 ], [ 0 .. 1 ] ]
---------
Level 0:

 * *
 * *
---------
Level 1:

 * *
 * *
---------
Level 2:

 s s
 . s
gap> filt := FiltrationBySpectralSequence( II_E );
<A descending filtration with degrees [ -1 .. 0 ] and graded parts:
```

```
-1:   <A non-zero cyclic torsion right module on a cyclic generator satisfying
     yet unknown relations>
   0:   <A rank 1 right module on 3 generators satisfying 2 relations>
of
<A right module on 4 generators satisfying yet unknown relations>>
gap> ByASmallerPresentation( filt );
<A descending filtration with degrees [ -1 .. 0 ] and graded parts:

-1:   <A non-zero cyclic torsion right module on a cyclic generator satisfying 1\
 relation>
   0:   <A rank 1 right module on 2 generators satisfying 1 relation>
of
<A rank 1 right module on 3 generators satisfying 2 relations>>
gap> Display( filt );
Degree -1:

Z/< 3 >
----------
Degree 0:

Z/< 3 > + Z^(1 x 1)
gap> Display( homMM );
Z/< 3 > + Z/< 3 > + Z^(1 x 1)
```

10.4.9 Functor_TensorProduct

▷ Functor_TensorProduct (global variable)

The tensor product bifunctor.

Code

```
InstallValue( Functor_TensorProduct_for_fp_modules,
        CreateHomalgFunctor(
                [ "name", "TensorProduct" ],
                [ "category", HOMALG_MODULES.category ],
                [ "operation", "TensorProductOp" ],
                [ "number_of_arguments", 2 ],
                [ "1", [ [ "covariant", "left adjoint", "distinguished" ] ] ],
                [ "2", [ [ "covariant", "left adjoint" ] ] ],
                [ "OnObjects", _Functor_TensorProduct_OnModules ],
                [ "OnMorphisms", _Functor_TensorProduct_OnMaps ],
                [ "MorphismConstructor", HOMALG_MODULES.category.MorphismConstructor ]
                )
        );
```

10.4.10 TensorProduct

▷ TensorProduct(*o1*, *o2*) (operation)
▷ *(*o1*, *o2*) (operation)

`o1` resp. `o2` could be a module, a map, a complex (of modules or of again of complexes), or a chain morphism.

The symbol `*` is a shorthand for several operations associated with the functor `Functor_TensorProduct_for_fp_modules` installed under the name `TensorProduct`.

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> M := HomalgMatrix( "[ 2, 3, 4,   5, 6, 7 ]", 2, 3, ZZ );
<A 2 x 3 matrix over an internal ring>
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 3 generators>
gap> N := HomalgMatrix( "[ 2, 3, 4, 5,   6, 7, 8, 9 ]", 2, 4, ZZ );
<A 2 x 4 matrix over an internal ring>
gap> N := LeftPresentation( N );
<A non-torsion left module presented by 2 relations for 4 generators>
gap> mat := HomalgMatrix( "[ \
> 1, 0, -3, -6, \
> 0, 1,  6, 11, \
> 1, 0, -3, -6  \
> ]", 3, 4, ZZ );
<A 3 x 4 matrix over an internal ring>
gap> phi := HomalgMap( mat, M, N );
<A "homomorphism" of left modules>
gap> IsMorphism( phi );
true
gap> phi;
<A homomorphism of left modules>
gap> L := Hom( ZZ, M );
<A rank 1 right module on 3 generators satisfying yet unknown relations>
gap> ByASmallerPresentation( L );
<A rank 1 right module on 2 generators satisfying 1 relation>
gap> Display( L );
Z/< 3 > + Z^(1 x 1)
gap> L;
<A rank 1 right module on 2 generators satisfying 1 relation>
gap> psi := phi * L;
<A homomorphism of right modules>
gap> ByASmallerPresentation( psi );
<A non-zero homomorphism of right modules>
gap> Display( psi );
[ [   0,   0,   1,   1 ],
  [   0,   0,   8,   1 ],
  [   0,   0,   0,  -2 ],
  [   0,   0,   0,   2 ] ]

the map is currently represented by the above 4 x 4 matrix
gap> ML := Source( psi );
<A rank 1 right module on 4 generators satisfying 3 relations>
gap> IsIdenticalObj( ML, M * L ); ## the caching at work
true
gap> NL := Range( psi );
<A rank 2 right module on 4 generators satisfying 2 relations>
gap> IsIdenticalObj( NL, N * L ); ## the caching at work
```

```
true
gap> Display( ML );
Z/< 3 > + Z/< 3 > + Z/< 3 > + Z^(1 x 1)
gap> Display( NL );
Z/< 3 > + Z/< 12 > + Z^(2 x 1)
```

Now we compute a certain natural filtration on the tensor product $M*L$:

Example

```
gap> P := Resolution( M );
<A non-zero right acyclic complex containing a single morphism of left modules\
 at degrees [ 0 .. 1 ]>
gap> GP := Hom( P );
<A non-zero acyclic cocomplex containing a single morphism of right modules at\
 degrees [ 0 .. 1 ]>
gap> CE := Resolution( GP );
<An acyclic cocomplex containing a single morphism of right complexes at degre\
es [ 0 .. 1 ]>
gap> FCE := Hom( CE, L );
<A non-zero acyclic complex containing a single morphism of left cocomplexes a\
t degrees [ 0 .. 1 ]>
gap> BC := HomalgBicomplex( FCE );
<A non-zero bicomplex containing left modules at bidegrees [ 0 .. 1 ]x
[ -1 .. 0 ]>
gap> II_E := SecondSpectralSequenceWithFiltration( BC );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 2 ] each consisting of left modules at bidegrees [ -1 .. 0 ]x
[ 0 .. 1 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 1 ], [ -1 .. 0 ] ]
---------
Level 0:

 * *
 * *
---------
Level 1:

 * *
 . .
---------
Level 2:

 s s
 . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -1 .. 0 ], [ 0 .. 1 ] ]
---------
```

```
Level 0:

 * *
 * *
---------
Level 1:

 * *
 . s
---------
Level 2:

 s s
 . s
gap> filt := FiltrationBySpectralSequence( II_E );
<An ascending filtration with degrees [ -1 .. 0 ] and graded parts:
   0:   <A rank 1 left module presented by 1 relation for 2 generators>
  -1:   <A non-zero left module presented by 2 relations for 2 generators>
of
<A non-zero left module presented by 10 relations for 6 generators>>
gap> ByASmallerPresentation( filt );
<An ascending filtration with degrees [ -1 .. 0 ] and graded parts:
   0:   <A rank 1 left module presented by 1 relation for 2 generators>
  -1:   <A non-zero torsion left module presented by 2 relations
             for 2 generators>
of
<A rank 1 left module presented by 3 relations for 4 generators>>
gap> Display( filt );
Degree 0:

Z/< 3 > + Z^(1 x 1)
----------
Degree -1:

Z/< 3 > + Z/< 3 >
gap> Display( ML );
Z/< 3 > + Z/< 3 > + Z/< 3 > + Z^(1 x 1)
```

10.4.11 Functor_Ext

▷ `Functor_Ext` (global variable)

The bifunctor `Ext`.

Below is the only *specific* line of code used to define `Functor_Ext_for_fp_modules` and all the different operations `Ext` in homalg.

Code

```
RightSatelliteOfCofunctor( Functor_Hom_for_fp_modules, "Ext" );
```

10.4.12 Ext

▷ `Ext([c, ]o1, o2[, str])` (operation)

Compute the `c`-th extension object of `o1` with `o2` where `c` is a nonnegative integer and `o1` resp. `o2` could be a module, a map, a complex (of modules or of again of complexes), or a chain morphism. If `str`="a" then the (cohomologically) graded object $Ext^i($`o1`,`o2`$)$ for $0 \leq i \leq$ `c` is computed. If neither `c` nor `str` is specified then the cohomologically graded object $Ext^i($`o1`,`o2`$)$ for $0 \leq i \leq d$ is computed, where d is the length of the internally computed free resolution of `o1`.

Each generator of a module of extensions is displayed as a matrix of appropriate dimensions.

Example
```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> M := HomalgMatrix( "[ 2, 3, 4,   5, 6, 7 ]", 2, 3, ZZ );;
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 3 generators>
gap> N := TorsionObject( M );
<A cyclic torsion left module presented by yet unknown relations for a cyclic \
generator>
gap> iota := TorsionObjectEmb( M );
<A monomorphism of left modules>
gap> psi := Ext( 1, iota, N );
<A homomorphism of right modules>
gap> ByASmallerPresentation( psi );
<A non-zero homomorphism of right modules>
gap> Display( psi );
[ [  2 ] ]

the map is currently represented by the above 1 x 1 matrix
gap> extNN := Range( psi );
<A non-zero cyclic torsion right module on a cyclic generator satisfying 1 rel\
ation>
gap> IsIdenticalObj( extNN, Ext( 1, N, N ) ); ## the caching at work
true
gap> extMN := Source( psi );
<A non-zero cyclic torsion right module on a cyclic generator satisfying 1 rel\
ation>
gap> IsIdenticalObj( extMN, Ext( 1, M, N ) ); ## the caching at work
true
gap> Display( extNN );
Z/< 3 >
gap> Display( extMN );
Z/< 3 >
```

10.4.13 Functor_Tor

▷ `Functor_Tor` (global variable)

The bifunctor `Tor`.

Below is the only *specific* line of code used to define `Functor_Tor_for_fp_modules` and all the different operations `Tor` in homalg.

Code
```
LeftSatelliteOfFunctor( Functor_TensorProduct_for_fp_modules, "Tor" );
```

10.4.14 Tor

▷ `Tor([c, ]o1, o2[, str])` (operation)

Compute the `c`-th torsion object of `o1` with `o2` where `c` is a nonnegative integer and `o1` resp. `o2` could be a module, a map, a complex (of modules or of again of complexes), or a chain morphism. If `str`="a" then the (cohomologically) graded object Tor_i(`o1`,`o2`) for $0 \leq i \leq$ `c` is computed. If neither `c` nor `str` is specified then the cohomologically graded object Tor_i(`o1`,`o2`) for $0 \leq i \leq d$ is computed, where d is the length of the internally computed free resolution of `o1`.

Example
```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> M := HomalgMatrix( "[ 2, 3, 4,   5, 6, 7 ]", 2, 3, ZZ );;
gap> M := LeftPresentation( M );
<A non-torsion left module presented by 2 relations for 3 generators>
gap> N := TorsionObject( M );
<A cyclic torsion left module presented by yet unknown relations for a cyclic \
generator>
gap> iota := TorsionObjectEmb( M );
<A monomorphism of left modules>
gap> psi := Tor( 1, iota, N );
<A homomorphism of left modules>
gap> ByASmallerPresentation( psi );
<A non-zero homomorphism of left modules>
gap> Display( psi );
[ [  1 ] ]

the map is currently represented by the above 1 x 1 matrix
gap> torNN := Source( psi );
<A non-zero cyclic torsion left module presented by 1 relation for a cyclic ge\
nerator>
gap> IsIdenticalObj( torNN, Tor( 1, N, N ) ); ## the caching at work
true
gap> torMN := Range( psi );
<A non-zero cyclic torsion left module presented by 1 relation for a cyclic ge\
nerator>
gap> IsIdenticalObj( torMN, Tor( 1, M, N ) ); ## the caching at work
true
gap> Display( torNN );
Z/< 3 >
gap> Display( torMN );
Z/< 3 >
```

10.4.15 Functor_RHom

▷ `Functor_RHom` (global variable)

The bifunctor RHom.

Below is the only *specific* line of code used to define Functor_RHom_for_fp_modules and all the different operations RHom in homalg.

```
RightDerivedCofunctor( Functor_Hom_for_fp_modules );
```

10.4.16 RHom

▷ RHom([c,]o1, o2[, str]) (operation)

Compute the c-th extension object of o1 with o2 where c is a nonnegative integer and o1 resp. o2 could be a module, a map, a complex (of modules or of again of complexes), or a chain morphism. The string str may take different values:

- If str="a" then $R^iHom(o1,o2)$ for $0 \leq i \leq c$ is computed.
- If str="c" then the c-th connecting homomorphism with respect to the short exact sequence o1 is computed.
- If str="t" then the exact triangle upto cohomological degree c with respect to the short exact sequence o1 is computed.

If neither c nor str is specified then the cohomologically graded object $R^iHom(o1,o2)$ for $0 \leq i \leq d$ is computed, where d is the length of the internally computed free resolution of o1.

Each generator of a module of derived homomorphisms is displayed as a matrix of appropriate dimensions.

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> m := HomalgMatrix( [ [ 8, 0 ], [ 0, 2 ] ], ZZ );;
gap> M := LeftPresentation( m );
<A left module presented by 2 relations for 2 generators>
gap> Display( M );
Z/< 8 > + Z/< 2 >
gap> M;
<A torsion left module presented by 2 relations for 2 generators>
gap> a := HomalgMatrix( [ [ 2, 0 ] ], ZZ );;
gap> alpha := HomalgMap( a, "free", M );
<A homomorphism of left modules>
gap> pi := CokernelEpi( alpha );
<An epimorphism of left modules>
gap> Display( pi );
[ [  1,  0 ],
  [  0,  1 ] ]

the map is currently represented by the above 2 x 2 matrix
gap> iota := KernelEmb( pi );
<A monomorphism of left modules>
gap> Display( iota );
[ [  2,  0 ] ]
```

```
the map is currently represented by the above 1 x 2 matrix
gap> N := Kernel( pi );
<A cyclic torsion left module presented by yet unknown relations for a cyclic \
generator>
gap> Display( N );
Z/< 4 >
gap> C := HomalgComplex( pi );
<A left acyclic complex containing a single morphism of left modules at degree\
s [ 0 .. 1 ]>
gap> Add( C, iota );
gap> ByASmallerPresentation( C );
<A non-zero short exact sequence containing
2 morphisms of left modules at degrees [ 0 .. 2 ]>
gap> Display( C );
-------------------------
at homology degree: 2
Z/< 4 >
-------------------------
[ [  0,  6 ] ]

the map is currently represented by the above 1 x 2 matrix
------------v------------
at homology degree: 1
Z/< 2 > + Z/< 8 >
-------------------------
[ [  0,  1 ],
  [  1,  1 ] ]

the map is currently represented by the above 2 x 2 matrix
------------v------------
at homology degree: 0
Z/< 2 > + Z/< 2 >
-------------------------
gap> T := RHom( C, N );
<An exact cotriangle containing 3 morphisms of right cocomplexes at degrees
[ 0, 1, 2, 0 ]>
gap> ByASmallerPresentation( T );
<A non-zero exact cotriangle containing
3 morphisms of right cocomplexes at degrees [ 0, 1, 2, 0 ]>
gap> L := LongSequence( T );
<A cosequence containing 5 morphisms of right modules at degrees [ 0 .. 5 ]>
gap> Display( L );
-------------------------
at cohomology degree: 5
Z/< 4 >
------------^------------
[ [  0,  3 ] ]

the map is currently represented by the above 1 x 2 matrix
-------------------------
at cohomology degree: 4
Z/< 2 > + Z/< 4 >
```

```
------------^------------
[ [  0,  1 ],
  [  0,  0 ] ]

the map is currently represented by the above 2 x 2 matrix
-------------------------
at cohomology degree: 3
Z/< 2 > + Z/< 2 >
------------^------------
[ [  1 ],
  [  0 ] ]

the map is currently represented by the above 2 x 1 matrix
-------------------------
at cohomology degree: 2
Z/< 4 >
------------^------------
[ [  0,  2 ] ]

the map is currently represented by the above 1 x 2 matrix
-------------------------
at cohomology degree: 1
Z/< 2 > + Z/< 4 >
------------^------------
[ [  0,  1 ],
  [  2,  0 ] ]

the map is currently represented by the above 2 x 2 matrix
-------------------------
at cohomology degree: 0
Z/< 2 > + Z/< 2 >
-------------------------
gap> IsExactSequence( L );
true
gap> L;
<An exact cosequence containing 5 morphisms of right modules at degrees
[ 0 .. 5 ]>
```

10.4.17 Functor_LTensorProduct

▷ `Functor_LTensorProduct` (global variable)

The bifunctor `LTensorProduct`.

Below is the only *specific* line of code used to define `Functor_LTensorProduct_for_fp_modules` and all the different operations `LTensorProduct` in homalg.

Code
```
LeftDerivedFunctor( Functor_TensorProduct_for_fp_modules );
```

10.4.18 LTensorProduct

▷ LTensorProduct([c,]o1, o2[, str]) (operation)

Compute the c-th torsion object of $o1$ with $o2$ where c is a nonnegative integer and $o1$ resp. $o2$ could be a module, a map, a complex (of modules or of again of complexes), or a chain morphism. The string str may take different values:

- If str="a" then $L_i TensorProduct(o1,o2)$ for $0 \leq i \leq c$ is computed.
- If str="c" then the c-th connecting homomorphism with respect to the short exact sequence $o1$ is computed.
- If str="t" then the exact triangle upto cohomological degree c with respect to the short exact sequence $o1$ is computed.

If neither c nor str is specified then the cohomologically graded object $L_i TensorProduct(o1,o2)$ for $0 \leq i \leq d$ is computed, where d is the length of the internally computed free resolution of $o1$.

Each generator of a module of derived homomorphisms is displayed as a matrix of appropriate dimensions.

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> m := HomalgMatrix( [ [ 8, 0 ], [ 0, 2 ] ], ZZ );;
gap> M := LeftPresentation( m );
<A left module presented by 2 relations for 2 generators>
gap> Display( M );
Z/< 8 > + Z/< 2 >
gap> M;
<A torsion left module presented by 2 relations for 2 generators>
gap> a := HomalgMatrix( [ [ 2, 0 ] ], ZZ );;
gap> alpha := HomalgMap( a, "free", M );
<A homomorphism of left modules>
gap> pi := CokernelEpi( alpha );
<An epimorphism of left modules>
gap> Display( pi );
[ [  1,  0 ],
  [  0,  1 ] ]

the map is currently represented by the above 2 x 2 matrix
gap> iota := KernelEmb( pi );
<A monomorphism of left modules>
gap> Display( iota );
[ [  2,  0 ] ]

the map is currently represented by the above 1 x 2 matrix
gap> N := Kernel( pi );
<A cyclic torsion left module presented by yet unknown relations for a cyclic \
generator>
gap> Display( N );
Z/< 4 >
gap> C := HomalgComplex( pi );
<A left acyclic complex containing a single morphism of left modules at degree\
```

```
s [ 0 .. 1 ]>
gap> Add( C, iota );
gap> ByASmallerPresentation( C );
<A non-zero short exact sequence containing
2 morphisms of left modules at degrees [ 0 .. 2 ]>
gap> Display( C );
-------------------------
at homology degree: 2
Z/< 4 >
-------------------------
[ [  0,  6 ] ]

the map is currently represented by the above 1 x 2 matrix
------------v------------
at homology degree: 1
Z/< 2 > + Z/< 8 >
-------------------------
[ [  0,  1 ],
  [  1,  1 ] ]

the map is currently represented by the above 2 x 2 matrix
------------v------------
at homology degree: 0
Z/< 2 > + Z/< 2 >
-------------------------
gap> T := LTensorProduct( C, N );
<An exact triangle containing 3 morphisms of left complexes at degrees
[ 1, 2, 3, 1 ]>
gap> ByASmallerPresentation( T );
<A non-zero exact triangle containing
3 morphisms of left complexes at degrees [ 1, 2, 3, 1 ]>
gap> L := LongSequence( T );
<A sequence containing 5 morphisms of left modules at degrees [ 0 .. 5 ]>
gap> Display( L );
-------------------------
at homology degree: 5
Z/< 4 >
-------------------------
[ [  1,  3 ] ]

the map is currently represented by the above 1 x 2 matrix
------------v------------
at homology degree: 4
Z/< 2 > + Z/< 4 >
-------------------------
[ [  0,  1 ],
  [  0,  1 ] ]

the map is currently represented by the above 2 x 2 matrix
------------v------------
at homology degree: 3
Z/< 2 > + Z/< 2 >
```

```
-------------------------
[ [  2 ],
  [  0 ] ]

the map is currently represented by the above 2 x 1 matrix
------------v------------
at homology degree: 2
Z/< 4 >
-------------------------
[ [  0,  2 ] ]

the map is currently represented by the above 1 x 2 matrix
------------v------------
at homology degree: 1
Z/< 2 > + Z/< 4 >
-------------------------
[ [  0,  1 ],
  [  1,  1 ] ]

the map is currently represented by the above 2 x 2 matrix
------------v------------
at homology degree: 0
Z/< 2 > + Z/< 2 >
-------------------------
gap> IsExactSequence( L );
true
gap> L;
<An exact sequence containing 5 morphisms of left modules at degrees
[ 0 .. 5 ]>
```

10.4.19 Functor_HomHom

▷ `Functor_HomHom` (global variable)

The bifunctor `HomHom`.

Below is the only *specific* line of code used to define `Functor_HomHom_for_fp_modules` and all the different operations `HomHom` in homalg.

```
Functor_Hom_for_fp_modules * Functor_Hom_for_fp_modules;
```

10.4.20 Functor_LHomHom

▷ `Functor_LHomHom` (global variable)

The bifunctor `LHomHom`.

Below is the only *specific* line of code used to define `Functor_LHomHom_for_fp_modules` and all the different operations `LHomHom` in homalg.

```
LeftDerivedFunctor( Functor_HomHom_for_fp_modules );
```

10.5 Tool Functors

10.6 Other Functors

10.7 Functors: Operations and Functions

Chapter 11

Symmetric Algebra and Koszul Complex

11.1 Symmetric Algebra: Constructor

11.1.1 SymmetricPower

▷ `SymmetricPower(k, M)` (operation)

Returns: a homalg module

Construct the k-th exterior power of module M.

11.2 Symmetric Algebra: Properties and Attributes

11.2.1 IsSymmetricPower

▷ `IsSymmetricPower(M)` (property)

Returns: `true` or `false`

Marks a module as an symmetric power of another module.

11.2.2 SymmetricPowerExponent

▷ `SymmetricPowerExponent(M)` (attribute)

Returns: an integer

The exponent of the symmetric power.

11.2.3 SymmetricPowerBaseModule

▷ `SymmetricPowerBaseModule(M)` (attribute)

Returns: a homalg module

The module that M is an symmetric power of.

Chapter 12

Exterior Algebra and Koszul Complex

What follows are several operations related to the exterior algebra of a free module:

- A constructor for the graded parts of the exterior algebra (“exterior powers”)
- Several Operations on elements of these exterior powers
- A constructor for the “Koszul complex”
- An implementation of the “Cayley determinant” as defined in [CQ11], which allows calculating greatest common divisors from finite free resolutions.

12.1 Exterior Algebra: Constructor

12.1.1 ExteriorPower

▷ `ExteriorPower(k, M)` (operation)

Returns: a homalg module

Construct the *k*-th exterior power of module *M*.

12.2 Exterior Algebra: Properties and Attributes

12.2.1 IsExteriorPower

▷ `IsExteriorPower(M)` (property)

Returns: `true` or `false`

Marks a module as an exterior power of another module.

12.2.2 ExteriorPowerExponent

▷ `ExteriorPowerExponent(M)` (attribute)

Returns: an integer

The exponent of the exterior power.

12.2.3 ExteriorPowerBaseModule

▷ `ExteriorPowerBaseModule(M)` (attribute)

Returns: a homalg module

The module that *M* is an exterior power of.

12.3 Exterior Algebra: Element Properties

12.3.1 IsExteriorPowerElement

▷ `IsExteriorPowerElement(x)` (property)

Returns: `true` or `false`

Checks if the element *x* is from an exterior power.

12.4 Exterior Algebra: Element Operations

12.4.1 Wedge (for elements of exterior powers of free modules)

▷ `Wedge(x, y)` (operation)

Returns: an element of an exterior power

Calculate $x \wedge y$.

12.4.2 ExteriorPowerElementDual

▷ `ExteriorPowerElementDual(x)` (operation)

Returns: an element of an exterior power

For *x* in a q-th exterior power of a free module of rank n, return $x*$ in the (n-q)-th exterior power, as defined in [CQ11].

12.4.3 SingleValueOfExteriorPowerElement

▷ `SingleValueOfExteriorPowerElement(x)` (operation)

Returns: a ring element

For *x* in a highest exterior power, returns its single coordinate in the canonical basis; i.e. $[x]$ as defined in [CQ11].

12.5 Koszul complex and Cayley determinant

12.5.1 KoszulCocomplex

▷ `KoszulCocomplex(a, E)` (operation)

Returns: a homalg cocomplex

Calculate the *E*-valued Koszul complex of `a`.

12.5.2 CayleyDeterminant

▷ `CayleyDeterminant(C)` (operation)

Returns: a ring element

Calculate the Cayley determinant of the complex `C`, as defined in [CQ11].

12.5.3 Gcd_UsingCayleyDeterminant

▷ `Gcd_UsingCayleyDeterminant(x, y[, ...])` (function)

Returns: a ring element

Returns the greatest common divisor of the given ring elements, calculated using the Cayley determinant.

Chapter 13

Examples

13.1 ExtExt

This corresponds to Example B.2 in [Bar09].

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> imat := HomalgMatrix( "[ \
>   262,  -33,   75,  -40, \
>   682,  -86,  196, -104, \
>  1186, -151,  341, -180, \
> -1932,  248, -556,  292, \
>  1018, -127,  293, -156  \
> ]", 5, 4, ZZ );
<A 5 x 4 matrix over an internal ring>
gap> M := LeftPresentation( imat );
<A left module presented by 5 relations for 4 generators>
gap> N := Hom( ZZ, M );
<A rank 1 right module on 4 generators satisfying yet unknown relations>
gap> F := InsertObjectInMultiFunctor( Functor_Hom_for_fp_modules, 2, N, "TensorN" );
<The functor TensorN for f.p. modules and their maps over computable rings>
gap> G := LeftDualizingFunctor( ZZ );;
gap> II_E := GrothendieckSpectralSequence( F, G, M );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 2 ] each consisting of left modules at bidegrees [ -1 .. 0 ]x
[ 0 .. 1 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 1 ], [ -1 .. 0 ] ]
---------
Level 0:

 * *
 * *
---------
Level 1:

```

```
 * *
 . .
---------
Level 2:

 s s
 . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -1 .. 0 ], [ 0 .. 1 ] ]
---------
Level 0:

 * *
 * *
---------
Level 1:

 * *
 . s
---------
Level 2:

 s s
 . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<An ascending filtration with degrees [ -1 .. 0 ] and graded parts:
   0:   <A non-torsion left module presented by 3 relations for 4 generators>
  -1:   <A non-zero left module presented by 33 relations for 8 generators>
of
<A non-zero left module presented by 27 relations for 19 generators>>
gap> ByASmallerPresentation( filt );
<An ascending filtration with degrees [ -1 .. 0 ] and graded parts:
   0:   <A rank 1 left module presented by 2 relations for 3 generators>

-1:   <A non-zero torsion left module presented by 6 relations for 6 generators>
of
<A rank 1 left module presented by 8 relations for 9 generators>>
gap> m := IsomorphismOfFiltration( filt );
<A non-zero isomorphism of left modules>
```

13.2 Purity

This corresponds to Example B.3 in [Bar09].

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> imat := HomalgMatrix( "[ \
>   262,  -33,   75,  -40, \
```

```
>   682,  -86,  196, -104, \
>  1186, -151,  341, -180, \
> -1932,  248, -556,  292, \
>  1018, -127,  293, -156  \
> ]", 5, 4, ZZ );
<A 5 x 4 matrix over an internal ring>
gap> M := LeftPresentation( imat );
<A left module presented by 5 relations for 4 generators>
gap> filt := PurityFiltration( M );
<The ascending purity filtration with degrees [ -1 .. 0 ] and graded parts:
   0:   <A free left module of rank 1 on a free generator>

-1:   <A non-zero torsion left module presented by 2 relations for 2 generators>
of
<A non-pure rank 1 left module presented by 2 relations for 3 generators>>
gap> M;
<A non-pure rank 1 left module presented by 2 relations for 3 generators>
gap> II_E := SpectralSequence( filt );
<A stable homological spectral sequence with sheets at levels
[ 0 .. 2 ] each consisting of left modules at bidegrees [ -1 .. 0 ]x
[ 0 .. 1 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a homological spectral sequence at bidegrees
[ [ 0 .. 1 ], [ -1 .. 0 ] ]
---------
Level 0:

 * *
 * *
---------
Level 1:

 * *
 . .
---------
Level 2:

 s .
 . .

Now the spectral sequence of the bicomplex:

a homological spectral sequence at bidegrees
[ [ -1 .. 0 ], [ 0 .. 1 ] ]
---------
Level 0:

 * *
 * *
---------
```

```
Level 1:

 * *
 . s
---------
Level 2:

 s .
 . s
gap> m := IsomorphismOfFiltration( filt );
<A non-zero isomorphism of left modules>
gap> IsIdenticalObj( Range( m ), M );
true
gap> Source( m );
<A non-torsion left module presented by 2 relations for 3 generators (locked)>
gap> Display( last );
[ [   0,   2,   0 ],
  [   0,   0,  12 ] ]

Cokernel of the map

Z^(1x2) --> Z^(1x3),

currently represented by the above matrix
gap> Display( filt );
Degree 0:

Z^(1 x 1)
----------
Degree -1:

Z/< 2 > + Z/< 12 >
```

13.3 TorExt-Grothendieck

This corresponds to Example B.5 in [Bar09].

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> imat := HomalgMatrix( "[ \
>   262,  -33,   75,  -40, \
>   682,  -86,  196, -104, \
>  1186, -151,  341, -180, \
> -1932,  248, -556,  292, \
>  1018, -127,  293, -156  \
> ]", 5, 4, ZZ );
<A 5 x 4 matrix over an internal ring>
gap> M := LeftPresentation( imat );
<A left module presented by 5 relations for 4 generators>
gap> F := InsertObjectInMultiFunctor( Functor_TensorProduct_for_fp_modules, 2, M, "TensorM" );
<The functor TensorM for f.p. modules and their maps over computable rings>
```

```
gap> G := LeftDualizingFunctor( ZZ );;
gap> II_E := GrothendieckSpectralSequence( F, G, M );
<A stable cohomological spectral sequence with sheets at levels
[ 0 .. 2 ] each consisting of left modules at bidegrees [ -1 .. 0 ]x
[ 0 .. 1 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a cohomological spectral sequence at bidegrees
[ [ 0 .. 1 ], [ -1 .. 0 ] ]
---------
Level 0:

 * *
 * *
---------
Level 1:

 * *
 . .
---------
Level 2:

 s s
 . .

Now the spectral sequence of the bicomplex:

a cohomological spectral sequence at bidegrees
[ [ -1 .. 0 ], [ 0 .. 1 ] ]
---------
Level 0:

 * *
 * *
---------
Level 1:

 * *
 . s
---------
Level 2:

 s s
 . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<A descending filtration with degrees [ -1 .. 0 ] and graded parts:

-1:   <A non-zero left module presented by yet unknown relations for 9 generator\
s>

0:   <A non-zero left module presented by yet unknown relations for 4 generators\
```

```
>
of
<A left module presented by yet unknown relations for 29 generators>>
gap> ByASmallerPresentation( filt );
<A descending filtration with degrees [ -1 .. 0 ] and graded parts:
  -1:   <A non-zero torsion left module presented by 4 relations
             for 4 generators>
   0:   <A rank 1 left module presented by 2 relations for 3 generators>
of
<A rank 1 left module presented by 6 relations for 7 generators>>
gap> m := IsomorphismOfFiltration( filt );
<A non-zero isomorphism of left modules>
```

13.4 TorExt

This corresponds to Example B.6 in [Bar09].

Example

```
gap> ZZ := HomalgRingOfIntegers( );
Z
gap> imat := HomalgMatrix( "[ \
>   262,  -33,   75,  -40, \
>   682,  -86,  196, -104, \
>  1186, -151,  341, -180, \
> -1932,  248, -556,  292, \
>  1018, -127,  293, -156  \
> ]", 5, 4, ZZ );
<A 5 x 4 matrix over an internal ring>
gap> M := LeftPresentation( imat );
<A left module presented by 5 relations for 4 generators>
gap> P := Resolution( M );
<A non-zero right acyclic complex containing a single morphism of left modules\
 at degrees [ 0 .. 1 ]>
gap> GP := Hom( P );
<A non-zero acyclic cocomplex containing a single morphism of right modules at\
 degrees [ 0 .. 1 ]>
gap> FGP := GP * P;
<A non-zero acyclic cocomplex containing a single morphism of left complexes a\
t degrees [ 0 .. 1 ]>
gap> BC := HomalgBicomplex( FGP );
<A non-zero bicocomplex containing left modules at bidegrees [ 0 .. 1 ]x
[ -1 .. 0 ]>
gap> p_degrees := ObjectDegreesOfBicomplex( BC )[1];
[ 0, 1 ]
gap> II_E := SecondSpectralSequenceWithFiltration( BC, p_degrees );
<A stable cohomological spectral sequence with sheets at levels
[ 0 .. 2 ] each consisting of left modules at bidegrees [ -1 .. 0 ]x
[ 0 .. 1 ]>
gap> Display( II_E );
The associated transposed spectral sequence:

a cohomological spectral sequence at bidegrees
```

```
[ [ 0 .. 1 ], [ -1 .. 0 ] ]
---------
Level 0:

 * *
 * *
---------
Level 1:

 * *
 . .
---------
Level 2:

 s s
 . .

Now the spectral sequence of the bicomplex:

a cohomological spectral sequence at bidegrees
[ [ -1 .. 0 ], [ 0 .. 1 ] ]
---------
Level 0:

 * *
 * *
---------
Level 1:

 * *
 * *
---------
Level 2:

 s s
 . s
gap> filt := FiltrationBySpectralSequence( II_E, 0 );
<A descending filtration with degrees [ -1 .. 0 ] and graded parts:

-1:   <A non-zero torsion left module presented by yet unknown relations for
     10 generators>
   0:   <A rank 1 left module presented by 3 relations for 4 generators>
of
<A left module presented by yet unknown relations for 13 generators>>
gap> ByASmallerPresentation( filt );
<A descending filtration with degrees [ -1 .. 0 ] and graded parts:
  -1:   <A non-zero torsion left module presented by 4 relations
              for 4 generators>
   0:   <A rank 1 left module presented by 2 relations for 3 generators>
of
<A rank 1 left module presented by 6 relations for 7 generators>>
gap> m := IsomorphismOfFiltration( filt );
```

```
<A non-zero isomorphism of left modules>
```

Appendix A

The Mathematical Idea behind Modules

As finite dimensional constructions in linear algebra over a field k boil down to solving (in)homogeneous linear systems over k, the Gaussian algorithm makes the whole theory perfectly computable.

Hence, for homological algebra (viewed as linear algebra over general rings) to be computable one needs to find appropriate substitutes for the Gaussian algorithm, where finite dimensionality has to be replaced by finite generatedness.

Luckily such substitutes exist for many rings of interest. Beside the well-known Hermite normal form algorithm for principal ideal rings it turns out that appropriate generalizations of the classical Gröbner basis algorithm for polynomial rings provide the desired substitute for a wide class of commutative *and* noncommutative rings. Note that for noncommutative rings the above discussion has to be restricted to homological constructions leading to one-sided linear systems $XA = B$ resp. $AX = B$ ($\rightarrow$ 1.1.3).

Appendix B

Logic Subpackages

B.1 LIMOD: Logical Implications for Modules

B.2 LIHOM: Logical Implications for Homomorphisms of Modules

Appendix C

Overview of the Modules Package Source Code

C.1 Relations and Generators

Filename `.gd`/`.gi`	Content
`HomalgRingMap`	operations and methods ring maps
`HomalgRelations`	a set of relations
`SetsOfRelations`	several sets of relations
`HomalgGenerators`	a set of generators
`SetsOfGenerators`	several sets of generators

Table: *The homalg package files*

C.2 The Basic Objects

Filename `.gd/.gi`	Content
`HomalgModule`	modules allowing several presentations linked with transition matrices
`HomalgSubmodule`	submodules allowing several sets of generators
`HomalgMap`	maps allowing several presentations of their source and target
`HomalgFiltration`	filtration of a module
`HomalgComplex`	(co)complexes of modules or of (co)complexes
`HomalgChainMap`	chain maps of (co)complexes consisting of maps or chain maps
`HomalgBicomplex`	bicomplexes of modules or of (co)complexes
`HomalgBigradedObject`	(differential) bigraded modules
`HomalgSpectralSequence`	homological and cohomological spectral sequences
`HomalgDiagram`	Betti diagrams

Table: *The homalg package files (continued)*

C.3 The High Level Homological Algorithms

Filename `.gd/.gi`	Content
`Modules`	subfactors, resolutions, parameterizations, intersections, annihilators
`ToolFunctors`	zero map, composition, addition, substraction, stacking, augmentation, and post dividing maps
`BasicFunctors`	cokernel, image, kernel, tensor product, Hom, Ext, Tor, RHom, LTensorProduct, HomHom, LHomHom, BaseChange (preliminary)
`OtherFunctors`	direct sum

Table: *The homalg package files (continued)*

C.4 Logical Implications for homalg Objects

Filename `.gd`/`.gi`	Content
`LIMOD`	logical implications for modules
`LIHOM`	logical implications for module homomorphisms

Table: *The homalg package files (continued)*

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