

# NoCK

## Computing obstruction for the existence of compact Clifford-Klein form

1.5

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**Maciej Bocheński**

**Piotr Jastrzębski**

**Anna Szczepkowska**

**Aleksy Tralle**

**Artur Woike**

**Maciej Bocheński**

Email: [mabo@matman.uwm.edu.pl](mailto:mabo@matman.uwm.edu.pl)

**Piotr Jastrzębski**

Email: [piojas@matman.uwm.edu.pl](mailto:piojas@matman.uwm.edu.pl)

Homepage: <http://wmii.uwm.edu.pl/~piojas/>

**Anna Szczepkowska**

Email: [anna.szczepkowska@matman.uwm.edu.pl](mailto:anna.szczepkowska@matman.uwm.edu.pl)

**Aleksy Tralle**

Email: [tralle@matman.uwm.edu.pl](mailto:tralle@matman.uwm.edu.pl)

Homepage: <http://wmii.uwm.edu.pl/~tralle/>

**Artur Woike**

Email: [awoike@matman.uwm.edu.pl](mailto:awoike@matman.uwm.edu.pl)

Homepage: <http://wmii.uwm.edu.pl/~awoike>

## **Abstract**

In this package we develop functions for an algorithm designed to find homogeneous spaces of semisimple non-compact Lie groups which do not admit compact Clifford-Klein forms.

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# Chapter 1

## Notation

We use the notation and convention for real Lie algebras as is from CoReLG Package, [DFdG14].

Example

```
gap> G:=RealFormById( "E", 7,3);
<Lie algebra of dimension 133 over SqrtField>
gap> rankG:=Dimension(CartanSubalgebra(G));
7
gap> rankRG:=Dimension(CartanSubspace(G));
3
gap> dimG:=Dimension(G);
133
gap> P:=CartanDecomposition( G ).P;
<vector space over SqrtField, with 54 generators>
gap> dimPforG:=Dimension(P);
54
gap> K:=CartanDecomposition( G ).K;
<Lie algebra of dimension 79 over SqrtField>
gap> rankK:= Dimension(CartanSubalgebra(K));
7
gap> dimK:= Dimension(K);
79
```

Classification can be found in Table 9 in [OV90], p. 312-317.

## Chapter 2

# Obstruction for the existence of compact Clifford-Klein form

In this chapter we describe functions for algorithm from [BJS<sup>+</sup>].

### 2.1 Technical functions

#### 2.1.1 NonCompactDimension

▷ NonCompactDimension( $G$ ) (function)

For a real Lie algebra  $G$  constructed by the function *RealFormById* (from [DFdG14]), this function returns the non-compact dimension of  $G$  (dimension of a non-compact part in Cartan decomposition of  $G$ ).

Example

```
gap> G:=RealFormById("E",6,2); # E6(6)
<Lie algebra of dimension 78 over SqrtField>
gap> dG:=NonCompactDimension(G);
42
```

#### 2.1.2 PCoefficients

▷ PCoefficients( $type$ ,  $rank$ ) (function)

Let  $G$  be a compact connected Lie group of the type  $type$  and the rank  $rank$ . Let  $\Lambda P_G = \Lambda(y_1, \dots, y_l)$  be the exterior algebra over the spaces  $P_G$  of the primitive elements in  $H^*(G)$ . Denote the degrees as follows  $|y_j| = 2p_j - 1, j = 1, \dots, l$ . This function returns coefficients  $p_1, \dots, p_l$ .

Example

```
gap> PCoefficients("D",5);
[ 2, 4, 6, 8, 5 ]
```

#### 2.1.3 PCalculate

▷ PCalculate( $pi$ ,  $qi$ ) (function)

Here  $p_i = \{p_1, \dots, p_l\}$  and  $q_i = \{q_1, \dots, q_m\}$  are sets of coefficients ( $l \geq m$ ). This function returns the polynomial:  $P(t) = \prod_{j=m+1}^l (1 + t^{2p_j-1}) \prod_{i=1}^m (1 - t^{2p_i}) / (1 - t^{2q_i})$ .

Example

```
gap> PCalculate([4,2,3],[2,2]);
t^9+t^5+t^4+1
```

### 2.1.4 AllZeroDH

▷ AllZeroDH(*type*, *rank*, *id*)

(function)

Let  $G^C$  be a complex Lie algebra of the type *type* and the rank *rank*. Let  $G$  be a real form of  $G^C$  with the index *id* (see *RealFormsInformation*, [DFdG14]). This function returns the set of degrees of  $P(t)$  that have zero coefficients over all permutation (see Section 7 in [BJS<sup>+</sup>]).

Example

```
gap> AllZeroDH("F",4,2);
[ 1, 2, 3, 5, 6, 7, 9, 10, 11, 13, 14, 15, 17, 18, 19, 21, 22, 23, 25, 26, 27 ]
```

## Chapter 3

# Algorithm example

In this chapter we use additionally functions from the following packages: CoReLG [DFdG14] and SLA [dG]. We will show in detail the split case (for a non-split case you should use algorithm to generate regular subalgebras from [DFdG15]). For example, we take  $G = \mathfrak{e}_{6(6)}$  (tuple "E",6,2 in CoReLG notation). We calculate *AllZeroDH* on it.

Example

```
gap> AllZeroDH("E",6,2);
[ 1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21, 22, 23, 24, 27,
  28, 29, 30, 31, 32, 35, 36, 37, 38, 39, 40, 41 ]
```

We generate all regular subalgebras of complexification.

Example

```
gap> GC:=SimpleLieAlgebra("E",6,Rationals);;
gap> REG:=RegularSemisimpleSubalgebras(GC);;
gap> L0:=List( REG, SemiSimpleType );
[ "A1", "A1 A1", "A2 A1", "A4", "D5", "A4 A1", "A2 A1 A1", "A2 A1 A2", "A3 A1",
  "A1 A1 A1", "A2", "A3", "A5", "A2 A2", "D4", "A5 A1", "A3 A1 A1", "A1 A1 A1 A1",
  "A2 A2 A2" ]
```

For each subalgebras we take the split real form and calculate its non-compact dimension.

Example

```
gap> L0[4];
"A4"
gap> RealFormsInformation( "A", 4 );

There are 4 simple real forms with complexification A4
  1 is of type su(5), compact form
  2 - 3 are of type su(p,5-p) with 1 <= p <= 2
  4 is of type sl(5,R)
Index '0' returns the realification of A4

gap> G:=RealFormById("A",4,4);;
gap> NonCompactDimension( G );
14
```

Number 14 is in output of *AllZeroDH* function, so for  $\mathfrak{g} = \mathfrak{e}_{6(6)}$  and  $\mathfrak{h} = \mathfrak{sl}(5, \mathbb{R})$  corresponding homogeneous spaces  $G/H$  do not have compact Clifford–Klein forms.



## Example

```

gap> L0[5];
"D5"
gap> RealFormsInformation( "D", 5 );

There are 7 simple real forms with complexification D5
  1 is of type so(10), compact form
  2 - 3 are of type so(2p,10-2p) with 1 <= p <= 2
  4 is of type so*(10)
  5 is of type so(9,1)
  6 - 7 are of type so(2p+1,10-2p-1) with 1 <= p <= 2
Index '0' returns the realification of D5

gap> G:=RealFormById("D",5,7);;
gap> NonCompactDimension( G );
25

```

Number 25 is not in output of *AllZeroDH* function, so for  $\mathfrak{g} = e_{6(6)}$  and  $\mathfrak{h} = \mathfrak{so}(5,5)$  our algorithm does not provide a solution to the problem.

# References

- [BJS<sup>+</sup>] M. Bocheński, P. Jastrzębski, A. Szczepkowska, A. Tralle, and A. Woike. Semisimple subalgebras in simple lie algebras and a computational approach to the compact clifford-klein forms problem. <https://doi.org/10.1080/10586458.2018.1492475>. 5, 6
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